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ANALYSIS OF THE BRYLINSKI-KOSTANT MODEL FOR SPHERICAL MINIMAL REPRESENTATIONS

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Abstract *We revisit with another view point the construction by R. Brylinski and B. Kostant of minimal representations of simple Lie groups. We start from a pair (V, Q), where V is a complex vector space and Q a homogeneous polynomial of degree 4 on V. The manifold Ξ is an orbit of a covering of* $\mathrm{Conf}(V, Q)$, *the conformal group of the pair (V, Q), in a finite dimensional representation space. By a generalized Kantor-Koecher-Tits construction we obtain a complex simple Lie algebra $\mathfrak{g}$, and furthermore a real form $\mathfrak{g}_{\mathbb{R}}$. The connected and simply connected Lie group $G_{\mathbb{R}}$ with* $\mathrm{Lie}(G_{\mathbb{R}}) = \mathfrak{g}_{\mathbb{R}}$ *acts unitarily on a Hilbert space of holomorphic functions defined on the manifold Ξ.*

Key words: Minimal representation, Kantor-Koecher-Tits construction, Jordan algebra, Bernstein identity, Meijer G-function.

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Introduction. — The construction of a realization for the minimal unitary representation of a simple Lie group by using geometric quantization has been the topic of many papers during the last thirty years: [Rawnsley-Sternberg,1982], [Torasso,1983], and more recently [Kobayashi-Ørsted,2003]. In a series of papers R. Brylinski and B. Kostant have introduced and studied a geometric quantization of minimal nilpotent orbits

for simple real Lie groups which are not of Hermitian type: [Brylinski-Kostant,1994,1995,1997], [Brylinski, 1997,1998] . They have constructed the associated irreducible unitary representation on a Hilbert space of half forms on the minimal nilpotent orbit. This can be considered as a Fock model for the minimal representation. In this paper we revisit this construction with another point of view. We start from a pair (V,Q) where V is a complex vector space and Q is a homogeneous polynomial on V of degree 4. The structure group $\mathrm{Str}(V,Q)$, for which Q is a semi-invariant, is assumed to have a symmetric open orbit. The conformal group $\mathrm{Conf}(V,Q)$ consists of rational transformations of V whose differential belongs to $\mathrm{Str}(V,Q)$. The main geometric object is the orbit Ξ of Q under K, a covering of $\mathrm{Conf}(V,Q)$, on a space $\mathcal{W}$ of polynomials on V. Then, by a generalized Kantor-Koecher-Tits construction, starting from the Lie algebra $\mathfrak{k}$ of K, we obtain a simple Lie algebra $\mathfrak{g}$ such that the pair $(\mathfrak{g},\mathfrak{k})$ is non Hermitian. As a vector space $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$, with $\mathfrak{p}=\mathcal{W}$. The main point is to define a bracket

$$\mathfrak{p}\oplus\mathfrak{p}\to\mathfrak{k},\quad (X,Y)\mapsto[X,Y],$$

such that $\mathfrak{g}$ becomes a Lie algebra. The Lie algebra $\mathfrak{g}$ is 5-graded:

$$\mathfrak{g}=\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_{0}\oplus\mathfrak{g}_{1}\oplus\mathfrak{g}_{2}.$$

In the fourth part one defines a representation ρ of $\mathfrak{g}$ on the space $\mathcal{O}(\Xi)_{\mathrm{fin}}$ of polynomial functions on Ξ. In a first step one defines a representation of an $\mathfrak{sl}_2$-triple (E,F,H). It turns out that this is only possible under a condition T. In such a case one obtains an irreducible unitary representation of the connected and simply connected group $\widetilde{G}_{\mathbb{R}}$ whose Lie algebra is a real form of $\mathfrak{g}$. The representation is spherical. It is realized on a Hilbert space of holomorphic functions on Ξ. There is an explicit formula for the reproducing kernel of $\mathcal{H}$ involving a hypergeometric function ${}_1F_2$. Further the space $\mathcal{H}$ is a weighted Bergman space with a weight taking in general both positive and negative values.

If $Q=R^2$ or $Q=R^4$ where R is a semi-invariant, then by considering a covering of order 2 or 4 of the orbit Ξ, one can obtain one or 3 other unitary representations of $\widetilde{G}_{\mathbb{R}}$. They are not spherical. If the condition T is not satisfied, by a modified construction, one still obtains an irreducible representation of $\widetilde{G}_{\mathbb{R}}$ which is not spherical. This last point is the subject of a paper in preparation by the first author.

The construction of a Schrödinger model for the minimal representation of the group $O(p,q)$ is the subject of a recent book by T. Kobayashi and G. Mano [2008]. We should not wonder that there is a link between both models: the Fock and the Schrödinger models, and that there is an analogue of the Bargmann transform in this setting.

1. The conformal group and the representation κ. — Let V be a finite dimensional complex vector space and Q a homogeneous polynomial on V. Define

$$L = \mathrm{Str}(V,Q) = \{g \in GL(V) \mid \exists \gamma = \gamma(g), Q(g\cdot x) = \gamma(g)Q(x)\}.$$

Assume that there exists $e \in V$ such that

(1) The symmetric bilinear form

$$\langle x, y\rangle = -D_x D_y \log Q(e),$$

is non-degenerate.

(2) The orbit $\Omega = L \cdot e$ is open.

(3) The orbit $\Omega = L \cdot e$ is symmetric, i.e. the pair (L, L_0), with $L_0 = \{g \in L \mid g \cdot e = e\}$, is symmetric, which means that there is an involutive automorphism ν of L such that L_0 is open in $\{g \in L \mid \nu(g) = g\}$.

We will equip the vector space V with a Jordan algebra structure. The Lie algebra $\mathfrak{l} = \mathrm{Lie}(L)$ of $L = \mathrm{Str}(V,Q)$ decomposes into the $+1$ and -1 eigenspaces of the differential of ν : $\mathfrak{l} = \mathfrak{l}_0 + \mathfrak{q}$, where $\mathfrak{l}_0 = \{X \in \mathfrak{l} \mid X.e = e\} = \mathrm{Lie}(L_0)$. Since the orbit Ω is open, the map

$$\mathfrak{q} \to V, \quad X \mapsto X.e,$$

is a linear isomorphism. If $X \cdot e = x \quad (X \in \mathfrak{q}, x \in V)$ one writes $X = T_x$. The product on V is defined by

$$xy = T_x \cdot y = T_x \circ T_y \cdot e.$$

THEOREM 1.1. — *This product makes V into a semi-simple complex Jordan algebra:*

(J1) *For $x, y \in V, xy = yx$.*

(J2) *For $x, y \in V, x^2(xy) = x(x^2y)$.*

(J3) *The symmetric bilinear form $\langle ., .\rangle$ is associative:*

$$\langle xy, z\rangle = \langle x, yz\rangle.$$

Proof. (a) This product is commutative. In fact

$$xy - yx = [T_x, T_y] \cdot e = 0,$$

since $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{l}_0$.

(b) Let τ be the differential of γ at the identity element of L: for $X \in \mathfrak{l}$,

$$\tau(X) = \frac{d}{dt}\Big|_{t=0} \gamma(\exp tX).$$

Lemma 1.2.

$$\begin{aligned} &\text{(i)} \quad (D_x \log Q)(e) = \tau(T_x), \\ &\text{(ii)} \quad (D_x D_y \log Q)(e) = -\tau(T_{xy}), \\ &\text{(iii)} \quad (D_x D_y D_z \log Q)(e) = \tfrac{1}{2}\,\tau(T_{(xy)z}). \end{aligned}$$

The proof amounts to differentiating at e the relation

$$\log Q(\exp T_x \cdot e) = \tau(T_x) + \log Q(e),$$

up to third order. (See Exercise 5 in [Satake, 1980], p.38.) Hence, by (ii), $\langle x, y\rangle = \tau(T_{xy})$, and, by (iii), the symmetric bilinear form $\langle ., .\rangle$ is associative.

(c) Define the associator of three elements x, y, z in V by

$$[x, y, z] = x(zy) - (xz)y = [L(x), L(y)]z.$$

Identity (J2) can be written: $[x^2, y, x] = 0$ for all $x, y \in V$. It can be shown by following the proof of Theorem 8.5 in [Satake,1980], p.34, which is also the proof of Theorem III.3.1 in [Faraut-Koranyi,1994], p.50. $\square$

The Jordan algebra V is a direct sum of simple ideals:

$$V = \bigoplus_{i=1}^{s} V_i,$$

and

$$Q(x) = \prod_{i=1}^{s} \Delta_i(x_i)^{k_i} \quad (x = (x_1, \ldots, x_s)),$$

where Δ_i is the determinant polynomial of the simple Jordan algebra V_i and the k_i are positive integers. The degree of Q is equal to $\sum_{i=1}^{s} k_i r_i$, where r_i is the rank of V_i.

The conformal group $\mathrm{Conf}(V, Q)$ is the group of rational transformations g of V generated by: the translations $z \mapsto z + a$ $(a \in V)$, the dilations $z \mapsto \ell \cdot z$ $(\ell \in L)$, and the inversion $j : z \mapsto -z^{-1}$. A transformation $g \in \mathrm{Conf}(V, Q)$ is conformal in the sense that the differential $Dg(z)$ belongs to $L \in \mathrm{Str}(V, Q)$ at any point z where g is defined.

Let $\mathcal{W}$ be the space of polynomials on V generated by the translated $Q(z - a)$ of Q. We will define a representation κ on $\mathcal{W}$ of $\mathrm{Conf}(V, Q)$ or of a covering of order two of it.

Case 1

In case there exists a character χ of $\mathrm{Str}(V,Q)$ such that $\chi^2=\gamma$, then let $K=\mathrm{Conf}(V,Q)$. Define the cocycle

$$\mu(g,z)=\chi((Dg(z)^{-1}) \quad (g\in K,\ z\in V),$$

and the representation κ of K on $\mathcal{W}$,

$$(\kappa(g)p)(z)=\mu(g^{-1},z)p(g^{-1}\cdot z).$$

The function $\kappa(g)p$ belongs actually to $\mathcal{W}$. In fact the cocycle $\mu(g,z)$ is a polynomial in z of degree $\leq \deg Q$ and

$$\begin{aligned}(\kappa(\tau_a)p)(z)&=p(z-a) \quad (a\in V),\\ (\kappa(\ell)p)(z)&=\chi(\ell)p(\ell^{-1}\cdot z) \quad (\ell\in L),\\ (\kappa(j)p)(z)&=Q(z)p(-z^{-1}).\end{aligned}$$

Case 2

Otherwise the group K is defined as the set of pairs (g,μ) with $g\in\mathrm{Conf}(V,Q)$, and μ is a rational function on V such that

$$\mu(z)^2=\gamma(Dg(z))^{-1}.$$

We consider on K the product $(g_1,\mu_1)(g_2,\mu_2)=(g_1g_2,\mu_3)$ with $\mu_3(z)=\mu_1(g_2\cdot z)\mu_2(z)$. For $\tilde g=(g,\mu)\in K$, define $\mu(\tilde g,z):=\mu(z)$. Then $\mu(\tilde g,z)$ is a cocycle:

$$\mu(\tilde g_1\tilde g_2,z)=\mu(\tilde g_1,\tilde g_2\cdot z)\mu(\tilde g_2,z),$$

where $\tilde g\cdot z=g\cdot z$ by definition.

Proposition 1.3. — (i) *The map*

$$K\to\mathrm{Conf}(V,Q),\quad \tilde g=(g,\mu)\mapsto g$$

is a surjective group morphism.

(ii) *For $g\in K$, $\mu(g,z)$ is a polynomial in z of degree $\leq \deg Q$.*

Proof. It is clearly a group morphism. We will show that the image contains a set of generators of $\mathrm{Conf}(V,Q)$. If g is a translation, then $(g,1)$ and $(g,-1)$ are elements in K. If $g=\ell\in L$, then $Dg(z)=\ell$, and $(\ell,\alpha),(\ell,-\alpha)$, with $\alpha^2=\gamma(\ell)^{-1}$, are elements in K. If $g\cdot z=j(z):=-z^{-1}$, then $Dg(z)^{-1}=P(z)$, where $P(z)$ denotes the quadratic representation of the Jordan algebra V: $P(z)=2T_z^2-T_{z^2}$, and $\gamma(P(z))=Q(z)^2$. Then $(j,Q(z)),(j,Q(-z))$ are elements in K. ∎

Let $P_{\max}$ denote the preimage in K of the maximal parabolic subgroup $L \ltimes N \subset \mathrm{Conf}(V, Q)$, where N is the subgroup of ranslations. For $g \in P_{\max}$, $\mu(g, z)$ does not depend on z, and $\chi(g) = \mu(g^{-1}, z)$ is a character of $P_{\max}$. For $g = (\ell, \alpha)$ $(\ell \in L)$, $\chi(g)^2 = \gamma(\ell)$.

Observe that the inverse in K of $\sigma = (j, Q(z))$ is $\sigma^{-1} = (j, Q(-z))$. If K is connected, then K is a covering of order 2 of $\mathrm{Conf}(V, Q)$. If not, the identity component K_0 of K is homeomorphic to $\mathrm{Conf}(V, Q)$.

The representation κ of K on $\mathcal{W}$ is then given by

$$\big(\kappa(g)p\big)(z) = \mu(g^{-1}, z)p(g^{-1} \cdot z).$$

In particular

$$\begin{aligned} \big(\kappa(g)p\big)(z) &= \chi(g)p(g^{-1} \cdot z) \quad (g \in P_{\max}), \\ \big(\kappa(\sigma)p\big)(z) &= Q(-z)p(-z^{-1}). \end{aligned}$$

Hence $p_0 \equiv 1$ is a highest weight vector with respect and $Q = \kappa(\sigma)p_0$ is a lowest weight vector.

Example 1

If $V = \mathbb{C}$, $Q(z) = z^n$, then $\mathrm{Str}(V, Q) = \mathbb{C}^*$, $\gamma(\ell) = \ell^n$, and $\mathrm{Conf}(V, Q) \simeq PSL(2, \mathbb{C})$ is the group of fractional linear transformations

$$z \mapsto g \cdot z = \frac{az + b}{cz + d}, \text{ with } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}).$$

Furthermore

$$Dg(z) = \frac{1}{(cz + d)^2}, \ \gamma\big(Dg(z)^{-1}\big) = (cz + d)^{2n}, \ \mu(g, z) = (cz + d)^n.$$

Hence, if n is even, then $K = PSL(2, \mathbb{C})$, and, if n is odd, then $K = SL(2, \mathbb{C})$.

The space $\mathcal{W}$ is the space of polynomials of degree $\leq n$ in one variable. The representation κ of K on $\mathcal{W}$ is given by

$$\big(\kappa(g)p\big)(z) = (cz + d)^n p\Big(\frac{az + b}{cz + d}\Big), \text{ if } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Example 2

If $V = M(n, \mathbb{C})$, $Q(z) = \det z$, then $\mathrm{Str}(V, Q) = GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$, acting on V by

$$\ell \cdot z = \ell_1 z \ell_2^{-1} \quad \ell = (\ell_1, \ell_2).$$

Then $\gamma(\ell) = \det \ell_1 \det \ell_2^{-1}$, and γ is not the square of a character of $\mathrm{Str}(V, Q)$. Furthermore $\mathrm{Conf}(V, Q) = PSL(2n, \mathbb{C})$ is the group of the rational transformations

$$z \mapsto g \cdot z = (az+b)(cz+d)^{-1}, \text{ with } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2n, \mathbb{C}),$$

decomposed in $n \times n$-blocs. To determine the differential of such a transformation, let us write (assuming c to be invertible)

$$g \cdot z = (az+c)(cz+d)^{-1} = ac^{-1} - (ac^{-1}d - b)(cz+d)^{-1},$$

and we get

$$Dg(z)w = (ac^{-1}d - b)(cz+d)^{-1}cw(cz+d)^{-1}.$$

Notice that $Dg(z) \in \mathrm{Str}(V, Q)$:

$$Dg(z)w = \ell_1 w \ell_2^{-1}, \text{ with } \ell_1 = (ac^{-1}d - b)(cz+d)^{-1}c, \ \ell_2 = (cz+d).$$

Since $\det(ac^{-1}d - b) \det c = \det g = 1$,

$$\gamma\big(Dg(z)^{-1}\big) = \det(cz+d)^2.$$

It follows that $K = SL(2n, \mathbb{C})$, and $\mu(g, z) = \det(cz+d)$.

The space $\mathcal{W}$ is a space of polynomials of an $n \times n$ matrix variable, with degree $\leq n$. The representation κ of K on $\mathcal{W}$ is given by

$$\big(\kappa(g)p\big)(z) = \det(cz+d)p\big((az+b)(cz+d)^{-1}\big), \text{ if } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

2. The orbit Ξ, and the irreducible K-invariant Hilbert subspaces of $\mathcal{O}(\Xi)$. — Let Ξ be the K-orbit of Q in $\mathcal{W}$:

$$\Xi = \{\kappa(g)Q \mid g \in K\}.$$

Then Ξ is a conical variety. In fact, if $\xi = \kappa(g)Q$, then, for $\lambda \in \mathbb{C}^*$, $\lambda\xi = \kappa(g \circ h_t)Q$, where $h_t \cdot z = e^{-t}z$ ($t \in \mathbb{C}$) with $\lambda = e^{2t}$.

A polynomial $\xi \in \mathcal{W}$ can be written

$$\xi(v) = wQ(v) + \text{ terms of degree } < 4 \quad (w \in \mathbb{C}),$$

and $w = w(\xi)$ is a linear form on $\mathcal{W}$ which is invariant under the parabolic subgroup $P_{\max}$. The set $\Xi_0 = \{\xi \in \Xi \mid w(\xi) \neq 0\}$ is open and dense in Ξ. A polynomial $\xi \in \Xi_0$ can be written

$$\xi(v) = wQ(v-z) \quad (w \in \mathbb{C}^*, z \in V).$$

Hence we get a coordinate system $(w, z) \in \mathbb{C}^* \times V$ for Ξ_0.

Proposition 2.1. — *In this system, the action of K is given by*

$$\kappa(g) : (w, z) \mapsto \big(\mu(g, z)w, g \cdot z\big).$$

Observe that the orbit Ξ can be seen as a line bundle over the conformal compactification of V.

Proof. Recall that, for $\xi \in \Xi$,

$$\big(\kappa(g)\xi\big)(v) = \mu(g^{-1}, v)\xi(g^{-1} \cdot v),$$

and, if $\xi(v) = wQ(v - z)$, then

$$= \mu(g^{-1}, v)wQ(g^{-1} \cdot v - z) = \mu(g^{-1}, v)wQ(g^{-1} \cdot v - g^{-1}g \cdot z).$$

By Lemma 6.6 in [Faraut-Gindikin,1996],

$$\mu(g, z)\mu(g, z')Q(g \cdot z - g' \cdot z') = Q(z - z').$$

Therefore

$$(\kappa(g)\xi)(v) = \mu(g^{-1}, g \cdot z)^{-1}wQ(v - g.z) = \mu(g, z)wQ(v - g \cdot z),$$

by the cocycle property. $\square$

The group K acts on the space $\mathcal{O}(\Xi)$ of holomorphic functions on Ξ by:

$$\big(\pi(g)f\big)(\xi) = f\big(\kappa(g)^{-1}\xi\big).$$

If $\xi \in \Xi_0$, i.e. $\xi(v) = wQ(v-z)$, and $f \in \mathcal{O}(\Xi)$, we will write $f(\xi) = \phi(w, z)$ for the restriction of f to Ξ_0. In the coordinates (w, z), the representation π is given by

$$(\pi(g)\phi)(w, z) = \phi(\mu(g^{-1}, z)w, g^{-1} \cdot z).$$

Let $\mathcal{O}_m(\Xi)$ denote the space of holomorphic functions f on Ξ, homogeneous of degree $m \in \mathbb{Z}$:

$$f(\lambda\xi) = \lambda^m f(\xi) \quad (\lambda \in \mathbb{C}^*).$$

The space $\mathcal{O}_m(\Xi)$ is invariant under the representation π. If $f \in \mathcal{O}_m(\Xi)$, then its restriction ϕ to Ξ_0 can be written $\phi(w, z) = w^m\psi(z)$, where ψ is a holomorphic function on V. We will write $\tilde{\mathcal{O}}_m(V)$ for the space of the functions ψ corresponding to the functions $f \in \mathcal{O}_m(\Xi)$, and denote by $\tilde{\pi}_m$

the representation of K on $\tilde{\mathcal{O}}_m(V)$ corresponding to the restriction π_m of π to $\mathcal{O}_m(\Xi)$. The representation $\tilde{\pi}_m$ is given by

$$\big(\tilde{\pi}_m(g)\psi\big)(z) = \mu(g^{-1}, z)^m \psi(g^{-1} \cdot z).$$

Observe that $(\tilde{\pi}_m(\sigma)1)(z) = Q(-z)^m$.

THEOREM 2.2. — (i) $\mathcal{O}_m(\Xi) = \{0\}$ *for* $m < 0$.

(ii) *The space* $\mathcal{O}_m(\Xi)$ *is finite dimensional, and the representation* π_m *is irreducible.*

(iii) *The functions* ψ *in* $\tilde{\mathcal{O}}_m(V)$ *are polynomials.*

Proof. (i) Assume $\mathcal{O}_m(\Xi) \neq \{0\}$. Let $f \in \mathcal{O}_m(\Xi)$, $f \not\equiv 0$, and $\phi(w, z) = \psi(z)w^m$ its restriction to Ξ_0. Then ψ is holomorphic on V, and

$$\big(\tilde{\pi}_m(\sigma)\psi\big)(z) = Q(-z)^m \psi(-z^{-1}),$$

is holomorphic as well. We may assume $\psi(e) \neq 0$. The function $h(\zeta) = \psi(\zeta e) \quad (\zeta \in \mathbb{C})$ is holomorphic on $\mathbb{C}$,

$$h(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k,$$

together with the function

$$Q(\zeta e)^m \psi(-\frac{1}{\zeta}e) = \zeta^{mN} h(-\frac{1}{\zeta}) = \zeta^{mN} \sum_{k=0}^{\infty} a_k (-\frac{1}{\zeta})^k \quad (N = \deg Q).$$

It follows that $m \geq 0$, and that $a_k = 0$ for $k > mN$.

(ii) The subspace

$$\{f \in \mathcal{O}_m(\Xi) \mid \forall a \in V, \pi(\tau_a)f = f\}$$

reduces to the functions Cw^m, hence is one dimensional. By the theorem of the highest weight [Goodman,2008], it follows that $\mathcal{O}_m(\Xi)$ is finite dimensional and irreducible.

(iii) Furthermore it follows that the functions in $\mathcal{O}_m(\Xi)$ are of the form $w^m\psi(z)$, where ψ is a polynomial on V of degree $\leq m \cdot \deg Q$. ∎

We fix a Euclidean real form $V_{\mathbb{R}}$ of the complex Jordan algebra V, denote by $z \mapsto \bar{z}$ the conjugation of V with respect to $V_{\mathbb{R}}$, and then consider the involution $g \mapsto \bar{g}$ of $\mathrm{Conf}(V, Q)$ given by: $\bar{g} \cdot z = \overline{g \cdot \bar{z}}$. For $(g, \mu) \in K$ define

$$\overline{(g, \mu)} = (\bar{g}, \bar{\mu}), \text{ where } \bar{\mu}(z) = \overline{\mu(\bar{z})}.$$

The involution α defined by $\alpha(g) = \sigma \circ \bar{g} \circ \sigma^{-1}$ is a Cartan involution of K (see Proposition 1.1. in [Pevzner,2002]), and

$$K_{\mathbb{R}} := \{g \in K \mid \alpha(g) = g\}$$

is a compact real form of K.

Example 1.

If $V = \mathbb{C}$, $Q(z) = z^n$. Then $V_{\mathbb{R}} = \mathbb{R}$, and $z \mapsto \bar{z}$ is the usual conjugation. We saw that $K = PSL(2, \mathbb{C})$ if n is even, and $SL(2, \mathbb{C})$ if n is odd. For $g \in SL(2, \mathbb{C})$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get

$$\alpha(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

Hence $K_{\mathbb{R}} = PSU(2)$ if n is even, and $K_{\mathbb{R}} = SU(2)$ if n is odd.

Example 2.

If $V = M(n, \mathbb{C})$, $Q(z) = \det z$, then $V_{\mathbb{R}} = Herm(n, \mathbb{C})$ and the conjugation is $z \mapsto z^*$. We saw that $K = SL(2n, \mathbb{C})$. For $g \in SL(2n, \mathbb{C})$,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we get

$$\alpha(g) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}.$$

Hence $K_{\mathbb{R}} = SU(2n)$.

We will define on $\mathcal{O}_m(\Xi)$ a $K_{\mathbb{R}}$-invariant inner product. Define the subgroup K_0 of K as $K_0 = L$ in Case 1, and the preimage of L in Case 2, relatively to the covering map $K \to \mathrm{Conf}(V, Q)$, and also $(K_0)_{\mathbb{R}} = K_0 \cap K_{\mathbb{R}}$. The coset space $M = K_{\mathbb{R}}/(K_0)_{\mathbb{R}}$, is a compact Hermitian space and is the conformal compactification of V. There is on M a $K_{\mathbb{R}}$-invariant probability measure, for which $M \setminus V$ has measure 0. Its restriction m_0 to V is a probability measure with a density which

can be computed by using the decomposition of V into simple Jordan algebras.

Let $H(z,z')$ be the polynomial on $V \times V$, holomorphic in z, anti-holomorphic in z' such that

$$H(x,x) = Q(e+x^2) \quad (x \in V_{\mathbb{R}}).$$

Put $H(z) = H(z,z)$. If z is invertible, then $H(z) = Q(\bar{z})Q(\bar{z}^{-1}+z)$.

Proposition 2.3. — *For $g \in K_{\mathbb{R}}$,*

$$H(g\cdot z_1, g\cdot z_2)\mu(g,z_1)\overline{\mu(g,z_2)} = H(z_1,z_2),$$

and

$$H(g\cdot z)|\mu(g,z)|^2 = H(z).$$

Proof. Recall that an element $g \in K_{\mathbb{R}}$ satisfies $\sigma\circ\bar{g}\circ\sigma^{-1} = g$, or $\sigma\circ\bar{g} = g\circ\sigma$. Recall also the cocycle property: for $g_1, g_2 \in K$,

$$\mu(g_1g_2,z) = \mu(g_1, g_2\cdot z)\mu(g_2,z).$$

Since $\mu(\sigma,z) = Q(z)$, it follows that, for $g \in K_{\mathbb{R}}$,

$$\mu(g,\sigma\cdot z)Q(z) = Q(\bar{g}\cdot z)\mu(\bar{g},z). \tag{1}$$

By Lemma 6.6 in [Faraut-Gindikin,1996], for $g \in K$,

$$Q(g\cdot z_1 - g\cdot z_2)\mu(g,z_1)\mu(g,z_2) = Q(z_1-z_2). \tag{2}$$

For $g \in K_{\mathbb{R}}$,

$$\begin{aligned} H(g\cdot z_1, g\cdot z_2) &= Q(\bar{g}\cdot z_2)Q(g\cdot z_1 - \sigma\bar{g}\cdot\bar{z}_2) \\ &= Q(\bar{g}\cdot\bar{z}_2)Q(g\cdot z_1 - g\sigma\bar{z}_2), \end{aligned}$$

and, by (2),

$$= Q(\bar{g}\cdot\bar{z}_2)\mu(g,z_1)^{-1}\mu(g,\sigma\cdot\bar{z}_2)^{-1}Q(z_1-\sigma\cdot\bar{z}_2).$$

Finally, by (1),

$$= \mu(g,z_1)^{-1}\mu(\bar{g},\bar{z}_2)^{-1}H(z_1,z_2).$$

∎

We define the norm of a function $\psi \in \tilde{\mathcal{O}}_m(V)$ by

$$\|\psi\|_m^2 = \frac{1}{a_m}\int_V |\psi(z)|^2 H(z)^{-m} m_0(dz),$$

with

$$a_m = \int_V H(z)^{-m} m_0(dz).$$

PROPOSITION 2.4. — (i) *This norm is $K_{\mathbb{R}}$-invariant. Hence, $\tilde{\mathcal{O}}_m(V)$ is a Hilbert subspace of $\mathcal{O}(V)$.*

(ii) *The reproducing kernel of $\tilde{\mathcal{O}}_m(V)$ is given by*

$$\tilde{\mathcal{K}}_m(z, z') = H(z, z')^m.$$

Proof. (i) From Proposition 2.3 it follows that, for $g \in K_{\mathbb{R}}$,

$$\begin{aligned}
\|\tilde{\pi}_m(g^{-1})\psi\|_m^2 &= \frac{1}{a_m} \int_V |\mu(g, z)|^{2m} |\psi(g^{-1} \cdot z)|^2 H(z)^{-m} m_0(dz) \\
&= \frac{1}{a_m} \int_V |\psi(g^{-1} \cdot z)|^2 H(g^{-1} \cdot z)^{-m} m_0(dz) \\
&= \frac{1}{a_m} \int_V |\psi(z)|^2 H(z)^{-m} m_0(dz) = \|\psi\|_m^2.
\end{aligned}$$

(ii) There is a unique function $\psi_0 \in \tilde{\mathcal{O}}_m(V)$ such that, for $\psi \in \tilde{\mathcal{O}}_m(V)$,

$$(\psi \mid \psi_0) = \psi(0).$$

The function ψ_0 is K_0-invariant, therefore constant: $\psi_0(z) = C$. Taking $\psi = \psi_0$, one gets $C^2 = C$, hence $C = 1$. It means that, if $\tilde{\mathcal{K}}_m(z, z')$ denotes the reproducing kernel of $\tilde{\mathcal{O}}_m(V)$,

$$\tilde{\mathcal{K}}_m(z, 0) = \tilde{\mathcal{K}}_m(0, z') = 1.$$

Since $\tilde{\mathcal{K}}_m(z, z')$ and $H(z, z')$ satisfy the following invariance properties: for $g \in K_{\mathbb{R}}$,

$$\begin{aligned}
\tilde{\mathcal{K}}_m(g \cdot z, g \cdot z') \mu(g, z)^m \overline{\mu(g, z')}^m &= \tilde{\mathcal{K}}_m(z, z'), \\
H(g \cdot z, g \cdot z') \mu(g, z) \overline{\mu(g, z')} &= H(z, z'),
\end{aligned}$$

it follows that

$$\tilde{\mathcal{K}}_m(z, z') = H(z, z')^m. \qquad \square$$

Since $\mathcal{O}_m(\Xi)$ is isomorphic to $\tilde{\mathcal{O}}_m(V)$, the space $\mathcal{O}_m(\Xi)$ becomes an invariant Hilbert subspace of $\mathcal{O}(\Xi)$, with reproducing kernel

$$\mathcal{K}_m(\xi, \xi') = \Phi(\xi, \xi')^m,$$

where

$$\Phi(\xi,\xi')=H(z,z')w\overline{w'}\qquad (\xi=(w,z),\xi'=(w',z')).$$

THEOREM 2.5. — *The group $K_{\mathbb{R}}$ acts multiplicity free on $\mathcal{O}(\Xi)$. The irreducible $K_{\mathbb{R}}$-invariant subspaces of $\mathcal{O}(\Xi)$ are the spaces $\mathcal{O}_m(\Xi)$ ($m\in\mathbb{N}$). If $\mathcal{H}\subset\mathcal{O}(\Xi)$ is a $K_{\mathbb{R}}$-invariant Hilbert subspace, the reproducing kernel of $\mathcal{H}$ can be written*

$$\mathcal{K}(\xi,\xi')=\sum_{m=0}^{\infty}c_m\Phi(\xi,\xi')^m,$$

with $c_m\geq 0$, such that the series $\sum_{m=0}^{\infty}c_m\Phi(\xi,\xi')^m$ converges uniformly on compact subsets in Ξ.

This multiplicity free property means that $K_{\mathbb{R}}$ acts multiplicity free on every $K_{\mathbb{R}}$-invariant Hilbert space $\mathcal{H}\subset\mathcal{O}(\Xi)$.

Proof. The representation π of $K_{\mathbb{R}}$ on $\mathcal{O}(\Xi)$ commutes with the $\mathbb{C}^*$-action by dilations and the spaces $\mathcal{O}_m(\Xi)$ are irreducible, and mutually inequivalent. It follows that $K_{\mathbb{R}}$ acts multiplicity free. ∎

In case of a weighted Bergman space there is an integral formula for the numbers c_m. For a positive function $p(\xi)$ on Ξ, consider the subspace $\mathcal{H}\subset\mathcal{O}(\Xi)$ of functions ϕ such that

$$\|\phi\|^2=\int_{\mathbb{C}\times V}|\phi(w,z)|^2p(w,z)m(dw)m_0(dz)<\infty,$$

where $m(dw)$ denotes the Lebesgue measure on $\mathbb{C}$.

THEOREM 2.6. — *Let F be a positive function on $[0,\infty[$, and define*

$$p(w,z)=F(H(z)|w|^2)H(z).$$

(i) *Then $\mathcal{H}$ is $K_{\mathbb{R}}$-invariant.*
(ii) *If*

$$\phi(w,z)=\sum_{m=0}^{\infty}w^m\psi_m(z),$$

then

$$\|\phi\|^2=\sum_{m=0}^{\infty}\frac{1}{c_m}\|\psi_m\|_m^2,$$

with

$$\frac{1}{c_m}=\pi a_m\int_0^{\infty}F(u)u^m du.$$

(iii) *The reproducing kernel of $\mathcal{H}$ is given by*

$$\mathcal{K}(\xi,\xi') = \sum_{m=0}^{\infty} c_m \Phi(\xi,\xi')^m.$$

Proof. a) Observe first that the function defined on Ξ by

$$(w,z) \mapsto |w|^2 H(z),$$

is $K_{\mathbb{R}}$-invariant. In fact, for $g \in K$,

$$\kappa(g) : (w,g) \mapsto \big(\mu(g,z)w, g\cdot z\big),$$

and, by Propositiion 2.3, for $g \in K_{\mathbb{R}}$,

$$|\mu(g,z)|^2 H(g\cdot z) = H(z).$$

Furthermore the measure $h(z)m(dw)m_0(dz)$ is also invariant under $K_{\mathbb{R}}$. In fact, under the transformation $z = g\cdot z', w = \mu(g,z')w'$ $(g \in K_{\mathbb{R}})$, we get

$$\begin{aligned} H(z)m(dw)m_0(dz) &= H(g\cdot z')|\mu(g,z')|^2 m(dw')m_0(dz') \\ &= H(z')m(dw')m_0(dz'). \end{aligned}$$

b) Assume that $p(w,z) = F\big(H(z)|w|^2\big)H(z)$. Then

$$\|\pi(g)\phi\|^2 = \int_{\mathbb{C}\times V} |\phi(\mu(g^{-1},z)w, g^{-1}\cdot z)|^2 F\big(H(z)|w|^2\big)H(z)m(dw)m_0(dz).$$

We put

$$g^{-1}\cdot z = z' \quad , \quad \mu(g^{-1},z)w = w'.$$

By the invariance of the measure $H(z)m(dw)m_0(dz)$, we obtain

$$\|\pi(g)\phi\|^2 =$$
$$\int_{\mathbb{C}\times V} |\phi(w',z')|^2 F\big(H(g\cdot z')|\mu(g^{-1},g\cdot z')|^{-2}|w'|^2\big)H(z')m(dw')m_0(dz').$$

Furthermore

$$H(g\cdot z')|\mu(g^{-1},g\cdot z')|^{-2} = H(g\cdot z')|\mu(g,z')|^2 = H(z'),$$

and, finally, $\|\pi(g)\phi\| = \|\phi\|$.

c) If $\phi(w,z) = w^m \psi(z)$, then

$$\|\phi\|^2 = \int_{\mathbb{C}\times V} |w|^{2m} |\psi(z)|^2 F\big(H(z)|w|^2\big) H(z) m(dw) m_0(dz').$$

We put $w' = \sqrt{H(z)}w$, then

$$\begin{aligned}\|\phi\|^2 &= \int_{\mathbb{C}\times V} H(z)^{-m} |w'|^{2m} |\psi(z)|^2 F(|w'|^2) m(dw') m_0(dz') \\ &= a_m \|\psi\|_m^2 \int_{\mathbb{C}} F(|w'|^2)|w'|^{2m} m(dw') \\ &= a_m \|\psi\|_m^2 \pi \int_0^\infty F(u) u^m du.\end{aligned}$$ ∎

3. Decomposition into simple Jordan algebras. — Let us decompose the semi-simple Jordan algebra V into simple ideals:

$$V = \bigoplus_{i=1}^s V_i.$$

Denote by n_i and r_i the dimension and the rank of the simple Jordan algebra V_i, and Δ_i the determinant polynomial. Then

$$Q(z) = \prod_{i=1}^s \Delta_i(z_i)^{k_i}.$$

Let $H_i(z,z')$ be the polynomial on $V_i \times V_i$, holomorphic in z, antiholomorphic in z', such that

$$H_i(z,z') = \Delta_i(e_i + x^2) \quad \big(x \in (V_i)_{\mathbb{R}}\big),$$

and put $H_i(z) = H_i(z,z)$. The measure m_0 has a density with respect to the Lebesgue measure m on V:

$$m_0(dz) = \frac{1}{C_0} H_0(z) m(dz),$$

with

$$H_0(z) = \prod_{i=1}^s H_i(z_i)^{-2\frac{n_i}{r_i}},$$

$$C_0 = \int_V H_0(z) m(dz).$$

The Lebesgue measure m will be chosen such that $C_0 = 1$.

PROPOSITION 3.1. — (i) *The polynomial Q satisfies the following Bernstein identity*

$$Q\Big(\frac{\partial}{\partial z}\Big)Q(z)^\alpha = B(\alpha)Q(z)^{\alpha-1} \quad (z \in \mathbb{C}),$$

where the Bernstein polynomial B is given by

$$B(\alpha) = \prod_{i=1}^{s} b_i(k_i\alpha)b_i(k_i\alpha - 1)\dots b_i(k_i\alpha - k_i + 1),$$

and b_i is the Bernstein polynomial relative to the determinant polynomial Δ_i.

(ii) *Furthermore*

$$Q\Big(\frac{\partial}{\partial z}\Big)H(z)^\alpha = B(\alpha)\overline{Q(z)}H(z)^{\alpha-1}.$$

Proof. (i) The Bernstein identity for Q follows from Proposition VII.1.4 in [Faraut-Korányi,1994].

(ii) For z invertible

$$H(z) = Q(\bar{z})Q(\bar{z}^{-1} + z),$$

and then, by (i),

$$\begin{aligned} Q\Big(\frac{\partial}{\partial z}\Big)H(z)^\alpha &= Q(\bar{z})^\alpha B(\alpha)Q(\bar{z}^{-1} + z)^{\alpha-1} \\ &= Q(\bar{z})B(\alpha)H(z)^{\alpha-1}. \end{aligned}$$

□

Example 1

If $V = \mathbb{C}$, $Q(z) = z^n$, then

$$\Big(\frac{d}{dz}\Big)^n z^{n\alpha} = B(\alpha)z^{n(\alpha-1)},$$

with

$$B(\alpha) = n\alpha(n\alpha - 1)\dots(n\alpha - n + 1).$$

Example 2

If $V = M(n, \mathbb{C})$, $Q(z) = \det z$, then

$$\det\Big(\frac{\partial}{\partial z}\Big)(\det z)^\alpha = B(\alpha)(\det z)^{\alpha-1},$$

with

$$B(\alpha) = \alpha(\alpha+1)\dots(\alpha+n-1).$$

Recall that we have introduced the numbers

$$a_m = \int_V H(z)^{-m} m_0(dz).$$

PROPOSITION 3.2.

$$a_m = \prod_{i=1}^s \frac{\Gamma_{\Omega_i}(2\frac{n_i}{r_i})}{\Gamma_{\Omega_i}(\frac{n_i}{r_i})} \prod_{i=1}^s \frac{\Gamma_{\Omega_i}(mk_i + \frac{n_i}{r_i})}{\Gamma_{\Omega_i}(mk_i + 2\frac{n_i}{r_i})},$$

where Γ_{Ω_i} is the Gindikin gamma function of the symmetric cone Ω_i in the Euclidean Jordan algebra $(V_i)_\mathbb{R}$.

Proof. If the Jordan algebra V is simple and $Q = \Delta$, the determinant polynomial, by Proposition X.3.4 in [Faraut-Korányi,1994],

$$\begin{aligned} a_m &= \int_V H(z)^{-m} m_0(dz) = \frac{1}{C_0}\int_V H(z)^{-m-2\frac{n}{r}} m(dz) \\ &= C\int_\Omega \Delta(e+x)^{-m-2\frac{n}{r}} m(dx). \end{aligned}$$

By Exercice 4 of Chapter VII in [Faraut-Korányi,1994] we obtain

$$a_m = C' \frac{\Gamma_\Omega(m+\frac{n}{r})}{\Gamma_\Omega(m+2\frac{n}{r})}.$$

In the general case

$$a_m = \frac{1}{C_0}\prod_{i=1}^s \int_{V_i} H_i(z_i)^{-mk_i - 2\frac{n_i}{r_i}} m_i(dz_i),$$

and the formula of the proposition follows. ∎

4. Generalized Kantor–Koecher–Tits construction. — From now on, Q is assumed to be of degree 4. The group of dilations of V : $h_t \cdot z = e^{-t}z$ ($t \in \mathbb{C}$) is a one parameter subgroup of L, and $\chi(h_t) = e^{-2t}$. Put $h_t = \exp(tH)$. Then $\mathrm{ad}(H)$ defines a grading of the Lie algebra $\mathfrak{k}$ of K:

$$\mathfrak{k} = \mathfrak{k}_{-1} + \mathfrak{k}_0 + \mathfrak{k}_1,$$

with $\mathfrak{k}_j = \{X \in \mathfrak{k} \mid \mathrm{ad}(H)X = jX\}$, ($j = -1, 0, 1$). Notice that

$$\mathfrak{k}_{-1} = Lie(N) \simeq V, \quad \mathfrak{k}_0 = Lie(L), \quad \mathrm{Ad}(\sigma) : \mathfrak{k}_j \to \mathfrak{k}_{-j},$$

and also that H belongs to the centre $\mathfrak{z}(\mathfrak{k}_0)$ of $\mathfrak{k}_0$. The element H defines also a grading of $\mathfrak{p} := \mathcal{W}$:

$$\mathfrak{p} = \mathfrak{p}_{-2} + \mathfrak{p}_{-1} + \mathfrak{p}_0 + \mathfrak{p}_1 + \mathfrak{p}_2,$$

where

$$\mathfrak{p}_j = \{p \in \mathfrak{p} \mid d\kappa(H)p = jp\}$$

is the set of polynomials in $\mathfrak{p}$, homogeneous of degree $j+2$. The subspaces $\mathfrak{p}_j$ are invariant under K_0. Furthermore $\kappa(\sigma) : \mathfrak{p}_j \to \mathfrak{p}_{-j}$, and

$$\mathfrak{p}_{-2} = \mathbb{C}, \quad \mathfrak{p}_2 = \mathbb{C}\, Q, \quad \mathfrak{p}_{-1} \simeq V, \quad \mathfrak{p}_1 \simeq V.$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Put $E = Q$, $F = 1$.

Theorem 4.1. — *There exists on $\mathfrak{g}$ a unique Lie algebra structure such that:*

$$\begin{aligned} (i) &\quad [X, X'] = [X, X']_{\mathfrak{k}} \quad (X, X' \in \mathfrak{k}),\\ (ii) &\quad [X, p] = d\kappa(X)p \quad (X \in \mathfrak{k}, p \in \mathfrak{p}),\\ (iii) &\quad [E, F] = H. \end{aligned}$$

Proof. Observe that (E, F, H) is an $\mathfrak{sl}_2$-triple, and that H defines a grading of

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2,$$

with

$$\mathfrak{g}_{-2} = \mathfrak{p}_{-2}, \quad \mathfrak{g}_{-1} = \mathfrak{k}_{-1} + \mathfrak{p}_{-1}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0, \quad \mathfrak{g}_1 = \mathfrak{k}_1 + \mathfrak{p}_1, \quad \mathfrak{g}_2 = \mathfrak{p}_2.$$

It is possible to give a direct proof of Theorem 4.1 (see Theorem 3.1. in [Achab,2011]). It is also possible to see this statement as a special case of constructions of Lie algebras by Allison and Faulkner [1984]. We describe below this construction in our case.

a) *Cayley-Dickson process.*

Let $x \mapsto x^*$ denote the symmetry with respect to the one dimensional subspace $\mathbb{C}e$:

$$x^* = \tfrac{1}{2}\langle x, e\rangle - x.$$

Observe that

$$\langle x, e\rangle = \tau(T_x) = D_x \log Q(e), \quad \langle e, e\rangle = 4.$$

On the vector space $W = V \oplus V$, one defines an algebra structure: if $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$, then $z_1 z_2 = z = (x, y)$ with

$$x = x_1 x_2 - (y_1 y_2^*)^*, \quad y = x_1^* y_2 + (y_1^* x_2^*)^*,$$

and an involution

$$\bar{z} = \overline{(x, y)} = (x, -y^*).$$

This involution is an antiautomorphism: $\overline{z_1 z_2} = \bar{z}_2 \bar{z}_1$. For $a, b \in W$, one introduces the endomorphisms $V_{a,b}$ and T_a given by

$$\begin{aligned} V_{a,b} z &= \{a, b, z\} := (a\bar{b})z + (z\bar{b})a - (z\bar{a})b, \\ T_a z &= V_{a,e} z = az + z(a - \bar{a}). \end{aligned}$$

By Theorem 6.6 in [Allison-Faulkner, 1984] the algebra W is structurable. This means that, for $a, b, c, d \in W$,

$$[V_{a,b}, V_{c,d}] = V_{V_{a,b}c,d} - V_{c,V_{b,a}d}. \qquad (*)$$

Moreover the structurable algebra W is simple. By $(*)$, the vector space spanned by the endomorphisms $V_{a,b}$ $(a, b \in W)$ is a Lie algebra denoted by $Instrl(W)$. This algebra is the Lie algebra $\mathfrak{g}_0$ in the grading, and its subalgebra $\mathfrak{k}_0$ is the structure algebra of the Jordan algebra V. The space S of skew-Hermitian elements in W, $S = \{z \in W \mid \bar{z} = -z\}$, has dimension one. Its elements are proportionnal to $s_0 = (0, e)$. The subspace $\{(x, 0) \mid x \in V\}$ of W is identified to V, and any element $z = (x, y) \in W$ can be written $z = x + s_0 y$.

b) *Generalized Kantor-Koecher-Tits construction.*

One defines a bracket on the vector space

$$\mathcal{K}(W) = \tilde{S} \oplus \tilde{W} \oplus Instrl(W) \oplus W \oplus S,$$

where $\tilde{S}$ is a second copy of S, and $\tilde{W}$ of W. This construction is described in [Allison,1979], and, by Corollary 6 in that paper, $\mathcal{K}(W)$ is a simple Lie algebra. On the subspace $\mathcal{K}(V) = \tilde{V} \oplus \mathfrak{str}(V) \oplus V$, this construction agrees

with the classical Kantor-Koecher-Tits construction, which produces the Lie algebra $\mathfrak{k} = \mathfrak{k}_{-1} \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_1$. This algebra $\mathcal{K}(W)$ satisfies property (i): the restriction of the bracket of $\mathcal{K}(W)$ to $\mathcal{K}(V)$ coincides to the one of $\mathcal{K}(V)$. It satisfies (iii) as well: $[s_0, \tilde{s}_0] = I$, the identity of $End(W)$. It remains to check property (ii). This can be seen as a consequence of the theorem of the highest weight for irreducible finite dimensional representations of reductive Lie algebras. In fact, the representation $d\kappa$ of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible with highest weight vector Q, with respect to any Borel subalgebra $\mathfrak{b} \subset \mathfrak{k}_0 + \mathfrak{k}_1$:

- If $X \in \mathfrak{k}_1$, then $\mathrm{d}\kappa(X)Q = 0$.
- If $X \in \mathfrak{k}_0$, such that $d\gamma(X) = 0$, then $d\kappa(X)Q = 0$, and $d\kappa(H)Q = 2Q$.

On the other hand, for the bracket of $\mathcal{K}(W)$,

- If $u \in V, [u, s_0] = 0$.
- If $X \in \mathfrak{str}(V)$, such that $\mathrm{tr}(X) = 0$, then $[X, s_0] = 0$ and $[H, s_0] = 2s_0$.

It follows that the adjoint representation of $\mathcal{K}(V) = \tilde{V} \oplus \mathfrak{str}(V) \oplus V$ on

$$\tilde{S} \oplus \tilde{s}_0\tilde{V} \oplus T_W \oplus s_0V \oplus S,$$

where $T_W = \{T_w = V_{w,e} \mid w \in W\}$, agrees with the representation $d\kappa$ of $\mathfrak{k}$ on $\mathfrak{p}$. In the present case, $T_w = L(w) + \frac{1}{2}\langle v, e\rangle Id$, if $w = u + s_0 v$ $(u, v \in V)$.

On the vector space

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

with

$$\mathfrak{g}_1 = W, \quad \mathfrak{g}_{-1} = W, \quad \mathfrak{g}_2 = \mathbb{C}\,E, \quad \mathfrak{g}_{-2} = \mathbb{C}\,F, \quad \mathfrak{g}_0 = Instrl(W),$$

one defines a bracket satisfying the following properties:

(1) $\mathfrak{g}_1 + \mathfrak{g}_2$ is a Heisenberg Lie algebra:

$$\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2, \quad (w_1, w_2) \mapsto w_1\bar{w}_2 - w_2\bar{w}_1 = \psi(w_1, w_2)s_0.$$

The bilinear form ψ is skew symmetric, and $[w_1, w_2] = \psi(w_1, w_2)E$.

(2) $\mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathfrak{g}_0, \quad (w, \tilde{w}) \mapsto V_{w,\tilde{w}}$.

(3) $\mathfrak{g}_2 \times \mathfrak{g}_{-1} \to \mathfrak{g}_1, \quad (\lambda E, \tilde{w}) \mapsto \lambda\tilde{w}$. $\square$

We introduce now a real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ which will be considered in the sequel. In Section 2 we have considered the involution α of K given by

$$\alpha(g) = \sigma \circ \bar{g} \circ \sigma^{-1} \quad (g \in K),$$

and the compact real form $K_{\mathbb{R}}$ of K:

$$K_{\mathbb{R}} = \{g \in K \mid \alpha(g) = g\}.$$

Recall that $\mathfrak{p}$ has been defined as a space of polynomial functions on V. For $p \in \mathfrak{p}$, define

$$\bar{p} = \overline{p(\bar{z})},$$

and consider the antilinear involution β of $\mathfrak{p}$ given by

$$\beta(p) = \kappa(\sigma)\bar{p}.$$

Observe that $\beta(E) = F$. The involution β is related to the involution α of K by the relation

$$\kappa\big(\alpha(g)\big) \circ \beta = \beta \circ \kappa(g) \quad (g \in K).$$

Hence, for $g \in K_{\mathbb{R}}$, $\kappa(g) \circ \beta = \beta \circ \kappa(g)$. Define

$$\mathfrak{p}_{\mathbb{R}} = \{p \in \mathfrak{p} \mid \beta(p) = p\}.$$

The real subspace $\mathfrak{p}_{\mathbb{R}}$ is invariant under $K_{\mathbb{R}}$, and irreducible for that action. The space $\mathfrak{p}$, as a real vector space, decomposes under $K_{\mathbb{R}}$ into two irreducible subspaces

$$\mathfrak{p} = \mathfrak{p}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}.$$

One checks that $E + F \in \mathfrak{p}_{\mathbb{R}}$ (and hence $i(E - F)$ as well).

Let $\mathfrak{u}$ be a compact real form of $\mathfrak{g}$ such that $\mathfrak{k} \cap \mathfrak{u} = \mathfrak{k}_{\mathbb{R}}$, the Lie algebra of $K_{\mathbb{R}}$. Then $\mathfrak{p}$ decomposes as

$$\mathfrak{p} = \mathfrak{p} \cap (i\mathfrak{u}) \oplus \mathfrak{p} \cap \mathfrak{u}$$

into two irreducible $K_{\mathbb{R}}$-invariant real subspaces. Looking at the subalgebra $\mathfrak{g}^0$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ generated by the triple (E, F, H), one sees that $E + F \in \mathfrak{p} \cap (i\mathfrak{u})$. Therefore $\mathfrak{p}_{\mathbb{R}} = \mathfrak{p} \cap (i\mathfrak{u})$, and

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$$

is a Lie algebra, real form of $\mathfrak{g}$, and the above decomposition is a Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$.

For the table of next page we have used the notation:

$$\varphi_n(z) = z_1^2 + \cdots + z_n^2, \quad (z \in \mathbb{C}^n).$$

In case of an exceptional Lie algebra $\mathfrak{g}$, the real form $\mathfrak{g}_{\mathbb{R}}$ has been identified by computing the Cartan signature.

V	Q	$\mathfrak{k}$	$\mathfrak{g}$	$\mathfrak{g}_{\mathbb{R}}$
$\mathbb{C}^n$	$\varphi_n(z)^2$	$\mathfrak{so}(n+2,\mathbb{C})$	$\mathfrak{sl}(n + 2,\mathbb{C})$	$\mathfrak{sl}(n + 2,\mathbb{R})$
$\mathbb{C}^p \oplus \mathbb{C}^q$	$\varphi_p(z)\varphi_q(z')$	$\mathfrak{so}(p+2,\mathbb{C}) \oplus \mathfrak{so}(q+2,\mathbb{C})$	$\mathfrak{so}(p+q+4,\mathbb{C})$	$\mathfrak{so}(p+2,q+2)$
$Sym(4,\mathbb{C})$	$\det z$	$\mathfrak{sp}(8,\mathbb{C})$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(6)}$
$M(4,\mathbb{C})$	$\det z$	$\mathfrak{sl}(8,\mathbb{C})$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(7)}$
$Skew(8,\mathbb{C})$	$\mathrm{Pfaff}(z)$	$\mathfrak{so}(16,\mathbb{C})$	$\mathfrak{e}_8$	$\mathfrak{e}_{8(8)}$
$Sym(3,\mathbb{C}) \oplus \mathbb{C}$	$\det z \cdot z'$	$\mathfrak{sp}(6,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{f}_4$	$\mathfrak{f}_{4(4)}$
$M(3,\mathbb{C}) \oplus \mathbb{C}$	$\det z \cdot z'$	$\mathfrak{sl}(6,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{e}_6$	$\mathfrak{e}_{6(2)}$
$Skew(6,\mathbb{C}) \oplus \mathbb{C}$	$\mathrm{Pfaff}(z) \cdot z'$	$\mathfrak{so}(12,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{e}_7$	$\mathfrak{e}_{7(-5)}$
$Herm(3,\mathbb{O})_{\mathbb{C}} \oplus \mathbb{C}$	$\det z \cdot z'$	$\mathfrak{e}_7 \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{e}_8$	$\mathfrak{e}_{8(-24)}$
$\mathbb{C} \oplus \mathbb{C}$	$z^3 \cdot z'$	$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{g}_2$	$\mathfrak{g}_{2(2)}$

5. Representation of the generalized Kantor-Koecher-Tits Lie algebra. — Following the method of R. Brylinski and B. Kostant, we will construct a representation ρ of $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ on the space of finite sums

$$\mathcal{O}(\Xi)_{\text{fin}} = \sum_{m=0}^{\infty} \mathcal{O}_m(\Xi),$$

such that, for all $X \in \mathfrak{k}$, $\rho(X) = d\pi(X)$. We define first a representation ρ of the subalgebra generated by E, F, H, isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. In particular

$$\rho(H) = d\pi(H) = \frac{d}{dt}\Big|_{t=0} \pi(\exp tH).$$

Hence, for $\phi \in \mathcal{O}_m(\Xi)$, $\rho(H)\phi = (\mathcal{E} - 2m)\phi$, where $\mathcal{E}$ is the Euler operator

$$\mathcal{E}\phi(w, z) = \frac{d}{dt}\Big|_{t=0} \phi(w, e^t z).$$

One introduces two operators $\mathcal{M}$ and $\mathcal{D}$. The operator $\mathcal{M}$ is a multiplication operator:

$$(\mathcal{M}\phi)(w, z) = w\phi(w, z),$$

which maps $\mathcal{O}_m(\Xi)$ into $\mathcal{O}_{m+1}(\Xi)$, and $\mathcal{D}$ is a differential operator:

$$(\mathcal{D}\phi)(w, z) = \frac{1}{w}\left(Q\Big(\frac{\partial}{\partial z}\Big)\phi\right)(w, z),$$

which maps $\mathcal{O}_m(\Xi)$ into $\mathcal{O}_{m-1}(\Xi)$. (Recall that $\mathcal{O}_{-1}(\Xi) = \{0\}$.) We denote by $\mathcal{M}^\sigma$ and $\mathcal{D}^\sigma$ the conjugate operators:

$$\mathcal{M}^\sigma = \pi(\sigma)\mathcal{M}\pi(\sigma)^{-1}, \quad \mathcal{D}^\sigma = \pi(\sigma)\mathcal{D}\pi(\sigma)^{-1}.$$

Given a sequence $(\delta_m)_{m\in\mathbb{N}}$ one defines the diagonal operator δ on $\mathcal{O}(\Xi)_{fin}$ by

$$\delta\Big(\sum_m \phi_m\Big) = \sum_m \delta_m \phi_m,$$

and put

$$\begin{aligned}&\rho(F) = \mathcal{M} - \delta \circ \mathcal{D},\\ &\rho(E) = \pi(\sigma)\rho_0(F)\pi(\sigma)^{-1} = \mathcal{M}^\sigma - \delta \circ \mathcal{D}^\sigma.\end{aligned}$$

(Observe that, since $\deg Q = 4$, then Q is even, and $\sigma = \sigma^{-1}$.)

Lemma 5.1.

$$\begin{aligned}&[\rho(H), \rho(E)] = 2\rho(E),\\ &[\rho(H), \rho(F)] = -2\rho(F).\end{aligned}$$

Proof. Since

$$\rho(H)\mathcal{M} : \psi(z)w^m \mapsto (\mathcal{E} - 2(m+1))\psi(z)w^{m+1},$$
$$\mathcal{M}\rho(H) : \psi(z)w^m \mapsto (\mathcal{E} - 2m)\psi(z)w^{m+1},$$

one obtains $[\rho(H), \mathcal{M}] = -2\mathcal{M}$. Since

$$\rho(H)\delta\mathcal{D} : \psi(z)w^m \mapsto \delta_{m-1}(\mathcal{E} - 2(m-1))Q\Big(\frac{\partial}{\partial z}\Big)\psi(z)w^{m-1},$$
$$\delta\mathcal{D}\rho(H) : \psi(z)w^m \mapsto \delta_{m-1}Q\Big(\frac{\partial}{\partial z}\Big)(\mathcal{E} - 2m)\psi(z)w^{m-1},$$

and, by using the identity

$$[Q\Big(\frac{\partial}{\partial z}\Big), \mathcal{E}] = 4Q\Big(\frac{\partial}{\partial z}\Big),$$

one gets

$$[\rho(H), \delta\mathcal{D}] : \psi(z)w^m \mapsto 2\delta_{m-1}Q\Big(\frac{\partial}{\partial z}\Big)\psi(z)w^{m-1}.$$

Finally $[\rho(H), \rho(F)] = -2\rho(F)$. Since the operator δ commutes with $\pi(\sigma)$, and $\pi(\sigma)\rho(H)\pi(\sigma)^{-1} = -\rho(H)$, we get also $[\rho(H), \rho(E)] = 2\rho(E)$. ∎

Let $\mathbb{D}(V)^L$ denote the algebra of L-invariant differential operators on V. This algebra is commutative. If V is simple and $Q = \Delta$, the determinant polynomial, then $\mathbb{D}(V)^L$ is isomorphic to the algebra $\mathcal{P}(\mathbb{C}^r)^{\mathfrak{S}_r}$ of symmetric polynomials in r variables. The map

$$D \mapsto \gamma(D), \quad \mathbb{D}(V) \to \mathcal{P}(\mathbb{C}^r)^{\mathfrak{S}_r},$$

is the Harish-Chandra isomorphism (see Theorem XIV.1.7 in [Faraut-Korányi,1994]). In general V decomposes into simple ideals,

$$V = \bigoplus_{i=1}^{s} V_i,$$

and $\mathbb{D}(V)^L$ is isomorphic to the algebra

$$\prod_{i=1}^{s} \mathcal{P}(\mathbb{C}^{r_i})^{\mathfrak{S}_{r_i}}.$$

The isomorphism is given by

$$D \mapsto \gamma(D) = \big(\gamma_1(D), \ldots, \gamma_s(D)\big),$$

where γ_i is the isomorphism relative to the algebra V_i. For $D \in \mathbb{D}(V)^L$, we define the adjoint D^* by $D^* = J \circ D \circ J$, where $Jf(z) = f \circ j(z) = f(-z^{-1})$. Then $\gamma(D^*)(\lambda) = \gamma(D)(-\lambda)$. (See Proposition XIV.1.8 in [Faraut-Korányi,1994].)

In our setting we define the Maass operator $\mathbf{D}_\alpha$ as

$$D_\alpha = Q(z)^{1+\alpha} Q\Big(\frac{\partial}{\partial z}\Big) Q(z)^{-\alpha}.$$

It is L-invariant. We write

$$\gamma_\alpha(\lambda) = \gamma(D_\alpha)(\lambda).$$

If V is simple and $Q = \Delta$, then

$$\gamma_\alpha(\lambda) = \prod_{i=1}^{r} \Big(\lambda_j - \alpha + \tfrac{1}{2}\,(\frac{n}{r} - 1)\Big),$$

([Faraut-Korányi,1994], p.296). If V is simple and $Q = \Delta^k$, then

$$\begin{aligned}\mathbf{D}_\alpha &= \Delta^{k+k\alpha} \Delta\Big(\frac{\partial}{\partial z}\Big)^k \Delta(z)^{-k\alpha} \\ &= \prod_{j=1}^{k} \Delta^{k\alpha+k-j+1} \Delta\Big(\frac{\partial}{\partial z}\Big) \Delta^{-(k\alpha+k-j)},\end{aligned}$$

and

$$\gamma_\alpha(\lambda) = \prod_{j=1}^{r} \big[\lambda_j - k\alpha + \tfrac{1}{2}\,(\frac{n}{r} - 1)\big]_k.$$

(We have used the Pochhammer symbol $[a]_k = a(a-1)\ldots(a-k+1)$.)

Proposition 5.2. — *In general*

$$\gamma_\alpha(\lambda) = \prod_{i=1}^{s} \prod_{j=1}^{r_i} \big[\lambda_j^{(i)} - k_i\alpha + \tfrac{1}{2}\,(\frac{n_i}{r_i} - 1)\big]_{k_i},$$

for $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(s)})$, $\lambda^{(i)} \in \mathbb{C}^{r_i}$.

We say that the pair (V, Q) has property (T) if there is a constant η such that, for $X \in \mathfrak{l} = Lie(L)$,

$$\operatorname{Tr}(X) = \eta\tau(X).$$

In such a case, for $g \in L$,

$$\operatorname{Det}(g) = \gamma(g)^{\eta},$$

and, for $x \in V$,

$$\operatorname{Det}\big(P(x)\big) = Q(x)^{2\eta}.$$

Furthermore $Q(x)^{-\eta}m(dx)$ is an L-invariant measure on the symmetric cone $\Omega \subset V_{\mathbb{R}}$, and $H_0(z) = H(z)^{-2\eta}$.

Let $V = \oplus_{i=1}^{s} V_i$ be the decomposition of V into simple ideals. Property (T) is equivalent to the following: there is a constant η such that

$$\frac{n_i}{r_i} = \eta k_i \quad (i = 1, \ldots, s).$$

In fact, for $x \in V$,

$$\operatorname{Tr}(T_x) = \sum_{i=1}^{s} \frac{n_i}{r_i} \operatorname{tr}_i(x_i), \quad \tau(T_x) = \sum_{i=1}^{s} k_i \operatorname{tr}_i(x_i),$$

with $x = (x_1, \ldots, x_s)$, $x_i \in V_i$.

Property (T) is satisfied either if V is simple, or if $V = \mathbb{C}^p \oplus \mathbb{C}^p$, and

$$Q(z) = (z_1^2 + \cdots + z_p^2)(z_{p+1}^2 + \cdots + z_{2p}^2).$$

Hence we get the following cases with property (T):

(1) $V = \mathbb{C}^n$, $Q(z) = (z_1^2 + \cdots + z_n^2)^2$, and then

$$\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{C}), \quad \mathfrak{k} = \mathfrak{so}(n+2, \mathbb{C}).$$

(2) $V = \mathbb{C}^p \oplus \mathbb{C}^p$, and then

$$\mathfrak{g} = \mathfrak{so}(2p+4, \mathbb{C}), \quad \mathfrak{k} = \mathfrak{so}(p+2, \mathbb{C}) \oplus \mathfrak{so}(p+2, \mathbb{C}).$$

(3) V is simple of rank 4, and $Q = \Delta$, the determinant polynomial. Then

$$(\mathfrak{g}, \mathfrak{k}) = \big(\mathfrak{e}_6, \mathfrak{sp}(8, \mathbb{C})\big), \quad \big(\mathfrak{e}_7, \mathfrak{sl}(8, \mathbb{C})\big), \quad \big(\mathfrak{e}_8, \mathfrak{so}(16, \mathbb{C})\big).$$

Observe that the case $V = \mathbb{C}^2$, $Q(z_1, z_2) = (z_1 z_2)^2 = z_1^2 z_2^2$ belongs both to (1) and (2). This corresponds to the isomorphisms:

$$\mathfrak{sl}(4, \mathbb{C}) \simeq \mathfrak{so}(6, \mathbb{C}),\ \ \mathfrak{so}(4, \mathbb{C}) \simeq \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C}).$$

PROPOSITION 5.3. — *The subspaces $\mathcal{O}_m(\Xi)$ are invariant under $[\rho(E), \rho(F)]$, and the restriction of $[\rho(E), \rho(F)]$ to $\mathcal{O}_m(\Xi)$ commutes with the L-action:*

$$[\rho(E), \rho(F)] : \mathcal{O}_m(\Xi) \to \mathcal{O}_m(\Xi), \quad \psi(z)w^m \mapsto (P_m\psi)(z)w^m,$$

where P_m is an L-invariant differential operator on V of degree ≤ 4. It is given by

$$P_m = \delta_m(\mathbf{D}_{-1} - \mathbf{D}^*_{-m-1}) + \delta_{m-1}(\mathbf{D}^*_{-m} - \mathbf{D}_0).$$

Proof. Restricted to $\mathcal{O}_m(\Xi)$,

$$\mathcal{M}^\sigma \mathcal{D} = \mathbf{D}_0, \quad \mathcal{D}\mathcal{M}^\sigma = \mathbf{D}_{-1}, \quad \mathcal{M}\mathcal{D}^\sigma = \mathbf{D}^*_{-m}, \quad \mathcal{D}^\sigma \mathcal{M} = \mathbf{D}^*_{-m-1}.$$

It follows that the restriction of the operator $[\rho(E), \rho(F)]$ to $\mathcal{O}_m(\Xi)$ is given by

$$\begin{aligned}
[\rho(E), \rho(F)] &= [\mathcal{M}^\sigma - \delta \circ \mathcal{D}^\sigma, \mathcal{M} - \delta \circ \mathcal{D}] \\
&= [\mathcal{M}, \delta \circ \mathcal{D}^\sigma] + [\delta \circ \mathcal{D}, \mathcal{M}^\sigma] \\
&= \mathcal{M}\delta\mathcal{D}^\sigma - \delta\mathcal{D}^\sigma\mathcal{M} + \delta\mathcal{D}\mathcal{M}^\sigma - \mathcal{M}^\sigma\delta \circ \mathcal{D} \\
&= \delta_m(\mathcal{D}\mathcal{M}^\sigma - \mathcal{D}^\sigma\mathcal{M}) + \delta_{m-1}(\mathcal{M}\mathcal{D}^\sigma - \mathcal{M}^\sigma\mathcal{D}) \\
&= \delta_m(\mathbf{D}_{-1} - \mathbf{D}^*_{-m-1}) + \delta_{m-1}(\mathbf{D}^*_{-m} - \mathbf{D}_0).
\end{aligned}$$

∎

By the Harish-Chandra isomorphism the operator P_m corresponds to the polynomial $p_m = \gamma(P_m)$,

$$p_m(\lambda) = \delta_m\big(\gamma_{-1}(\lambda) - \gamma_{-m-1}(-\lambda)\big) + \delta_{m-1}\big(\gamma_{-m}(-\lambda) - \gamma_0(\lambda)\big).$$

The question is now whether it is possible to choose the sequence (δ_m) in such a way that $[\rho(E), \rho(F)] = \rho(H)$. Recall that restricted to $\mathcal{O}_m(\Xi)$,

$$\rho(H) = \mathcal{E} - 2m,$$

where $\mathcal{E}$ is the Euler operator

$$\mathcal{E}\phi(w, z) = \frac{d}{dt}\Big|_{t=0}\phi(w, e^t z).$$

Then, by Proposition 5.3, it amounts to checking that, for every m,

$$p_m(\lambda) = \gamma(\mathcal{E})(\lambda) - 2m.$$

THEOREM 5.4. — *It is possible to choose the sequence (δ_m) such that*

$$[\rho(H), \rho(E)] = 2\rho(E), \quad [\rho(H), \rho(F)] = -2\rho(F), \quad [\rho(E), \rho(F)] = \rho(H),$$

if and only if (V, Q) has property (T), and then

$$\delta_m = \frac{A}{(m+\eta)(m+\eta+1)},$$

where A is a constant depending on (V, Q).

Proof. a) Let us assume first that the Jordan algebra V is simple of rank 4. In such a case

$$\gamma_\alpha(\lambda) = \prod_{j=1}^{4} \Big(\lambda_j - \alpha + \tfrac{1}{2}(\eta - 1)\Big) \quad (\eta = \frac{n}{r})$$

(Proposition 5.2) . With $X_j = \lambda_j + \frac{1}{2}(\eta - 1)$, the polynomial p_m can be written

$$p_m(\lambda) = \delta_m \Big(\prod_{j=1}^{4}(X_j + 1) - \prod_{j=1}^{4}(X_j - m - \eta)\Big)$$
$$+ \delta_{m-1}\Big(\prod_{j=1}^{4}(X_j - m + 1 - \eta) - \prod_{j=1}^{4} X_j\Big).$$

Furthermore

$$\gamma(\mathcal{E})(\lambda) - 2m = \sum_{j=1}^{4} \lambda_j - 2m = \sum_{j=1}^{4} X_j - 2(m + \eta - 1).$$

LEMMA 5.5. — *The identity in the four variables X_j*

$$\alpha\Big(\prod_{j=1}^{4}(X_j + 1) - \prod_{j=1}^{4}(X_j - b_j - 1)\Big) + \beta\Big(\prod_{j=1}^{4}(X_j - b_j) - \prod_{j=1}^{4} X_j\Big)$$
$$= \sum_{j=1}^{4} X_j + c$$

holds if and only if there is a constant b such that

$$
\begin{aligned}
&b_1 = b_2 = b_3 = b_4 = b, \;\; c = -2b, \\
&\alpha = \frac{1}{(b+1)(b+2)}, \;\; \beta = \frac{1}{b(b+1)}.
\end{aligned}
$$

Hence we apply the lemma, and get $b = m + \eta - 1$. ∎

b) In the general case

$$
\begin{aligned}
\gamma_\alpha(\lambda) &= \prod_{i=1}^{s} \prod_{j=1}^{r_i} \left[\lambda_j^{(i)} - k_i\alpha + \tfrac{1}{2}\left(\frac{n_i}{r_i} - 1\right)\right]_{k_i} \\
&= \prod_{i=1}^{s} \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} \Big(\lambda_j^{(i)} - k_i\alpha + \tfrac{1}{2}\Big(\frac{n_i}{r_i} - 1\Big) - (k-1)\Big) \\
&= A \prod_{i=1}^{s} \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} \Big(\frac{\lambda_j^{(i)}}{k_i} - \alpha + \frac{1}{2k_i}\Big(\frac{n_i}{r_i} - 1\Big) - \frac{k-1}{k_i}\Big),
\end{aligned}
$$

with $A = \prod_{i=1}^{s} k_i^{k_i r_i}$. We introduce the notation

$$
\begin{aligned}
X_{jk}^{(i)} &= \frac{\lambda_j^{(i)}}{k_i} + \frac{1}{2k_i}\Big(\frac{n_i}{r_i} - 1\Big) - \frac{k-1}{k_i}, \\
b_m^{(i)} &= m + \frac{n_i}{k_i r_i} - 1.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
p_m(\lambda) =& A\delta_m \Big(\prod_{i=1}^{s} \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} (X_{jk}^{(i)} + 1) - \prod_{i=1}^{s} \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} (X_{jk}^{(i)} - b_m^{(i)} - 1)\Big) \\
&+ A\delta_{m-1} \Big(\prod_{i=1}^{s} \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} (X_{jk}^{(i)} - b_m^{(i)}) - \prod_{i=1}^{s} \prod_{j=1}^{r_i} \prod_{k=1}^{k_i} (X_{jk}^{(i)})\Big),
\end{aligned}
$$

and

$$
\gamma(\mathcal{E})(\lambda) = \sum_{i=1}^{s} \sum_{j=1}^{r_i} \sum_{k=1}^{k_i} X_{jk}^{(i)} - \tfrac{1}{2} \sum_{i=1}^{s} \sum_{j=1}^{r_i} \sum_{k=1}^{k_i} b_m^{(i)}.
$$

If the rank of V is equal to 4, then the k_i are equal to 1, and the four variables $X_{j1}^{(i)}$ are independant. By Lemma 5.5, Theorem 5.4 is proven in that case.

If the rank r of V is < 4, then

$$X^{(i)}_{jk} = X^{(i)}_{j1} - \frac{k-1}{k_i},$$

and there are only r independant variables: $X^{(i)}_{j1}$. In that case Theorem 5.4 is proven by using an alternative form of Lemma 5.5: $\square$

LEMMA 5.6. — *To a partition $k = (k_1, \dots, k_\ell)$ of 4 and length ℓ:*

$$k_1 + \ldots + k_\ell = 4,$$

and the numbers γ_{ij} $(1 \leq i \leq \ell,\ 1 \leq j \leq k_i - 1)$, one associates the polynomial F in the ℓ variables $T_1, \dots, T_\ell$:

$$F(T_1, \dots, T_\ell) = \prod_{i=1}^{\ell} T_i \prod_{j=1}^{k_i-1} (T_i + \gamma_{ij}).$$

Given $\alpha, \beta, c \in \mathbb{R}$, and $b_1, \dots b_\ell \in \mathbb{R}$, then

$$\begin{aligned} &\alpha\big(F(T_1+1, \dots, T_\ell+1) - F(T_1 - b_1 - 1, \dots, T_\ell - b_\ell - 1)\big) \\ &+ \beta\big(F(T_1 - b_1, \dots, T_\ell - b_\ell) - F(T_1, \dots, T_\ell\big) = \sum_{i=1}^{\ell} T_i + c \end{aligned}$$

is an identity in the variables $T_1, \dots, T_\ell$ if and only if there exists b such that

$$b_1 = \ldots = b_\ell = b,\ \alpha = \frac{1}{(b+1)(b+2)},\ \beta = \frac{1}{b(b+1)},$$

and

$$c = \sum_{i=1}^{\ell} \sum_{j=1}^{k_i-1} \gamma_{ij} - 2b.$$

For $p \in \mathfrak{p}$, define the multiplication operator $\mathcal{M}(p)$ given by

$$\big(\mathcal{M}(p)\phi\big)(w,z) = wp(z)\phi(w,z).$$

Observe that $\mathcal{M}(1) = \mathcal{M}$. Then, for $g \in K$,

$$\mathcal{M}\big(\kappa(g)p\big) = \pi(g)\mathcal{M}(p)\pi(g^{-1}).$$

In fact

$$\big(\mathcal{M}(p)\pi(g^{-1})\phi\big)(w,z) = wp(z)\phi\big(\mu(g,z)w, g\cdot z\big),$$

and

$$\begin{aligned}
&\big(\pi(g)\mathcal{M}(p)\pi(g^{-1})\phi\big)(w,z)\\
&=\mu(g^{-1},z)wp(g^{-1}\cdot z)\phi\big(\mu(g^{-1},z)\mu(g,g^{-1}\cdot z)w,g^{-1}g\cdot z\big)\\
&=w\big(\kappa(z)p\big)(z)\phi(w,z)=\mathcal{M}\big(\kappa(g)p\big)\phi(w,z).
\end{aligned}$$

PROPOSITION 5.7. — *There is a unique map*

$$\mathfrak{p}\to \mathrm{End}\big(\mathcal{O}_{\mathrm{fin}}(\Xi)\big),\quad p\mapsto \mathcal{D}(p),$$

such that $\mathcal{D}(1)=\mathcal{D}$, *and, for* $g\in K$,

$$\mathcal{D}\big(\kappa(g)p\big)=\pi(g)\mathcal{D}(p)\pi(g^{-1}).$$

Proof. Recall that, for $g\in P_{\max}$,

$$\big(\kappa(g)p\big)(z)=\chi(g)p(g^{-1}\cdot z),$$

and

$$\big(\pi(g)\phi\big)(w,z)=\phi\big(\chi(g)w,g^{-1}\cdot g\big).$$

Let us show that, for $g\in P_{\max}$,

$$\pi(g)\mathcal{D}\pi(g^{-1})=\chi(g)\mathcal{D}.$$

Observe first that, for $\ell\in L$ and a smooth function ψ on V,

$$Q\Big(\frac{\partial}{\partial z}\Big)\big(\psi(\ell\cdot z)\big)=\gamma(\ell)\Big(Q\big(\frac{\partial}{\partial z}\big)\psi\Big)(\ell\cdot z).$$

Therefore, for $g\in P_{\max}$,

$$\begin{aligned}
\mathcal{D}\pi(g^{-1})\phi(w,z)&=\frac{1}{w}Q\Big(\frac{\partial}{\partial z}\Big(\phi\big(\chi(g^{-1})w,g\cdot z\big)\Big)\\
&=\frac{1}{w}\chi(g)^2\Big(Q\big(\frac{\partial}{\partial z}\big)\phi\Big)\big(\chi(g^{-1})w,g\cdot z\big),
\end{aligned}$$

and

$$\big(\pi(g)\mathcal{D}\pi(g^{-1})\phi\big)(w,z)=\frac{1}{\chi(g)w}\chi(g)^2\Big(Q\big(\frac{\partial}{\partial z}\big)\phi\Big)(w,z)=\chi(g)\mathcal{D}\phi(w,z).$$

It follows that the vector subspace in $\mathrm{End}\big(\mathcal{O}_{\mathrm{fin}}(\Xi)\big)$ generated by the endomorphisms $\pi(g)\mathcal{D}\pi(g^{-1})$ $(g\in K)$ is a representation space for K

equivalent to $\mathfrak{p}$. (See Theorem 3.10 in [Brylinski-Kostant,1994].) Hence there exists a unique K-equivariant map $p \mapsto \mathcal{D}(p)$ such that $\mathcal{D}(1) = \mathcal{D}$.
$\square$

For $p \in \mathfrak{p}$, define

$$\rho(p) = \mathcal{M}(p) - \delta\mathcal{D}(p).$$

Observe that this definition is consistent with the definition of $\rho(E)$ and $\rho(F)$. Recall that, for $X \in \mathfrak{k}$, $\rho(X) = d\pi(X)$. Hence we get a map

$$\rho : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \to \mathrm{End}\big(\mathcal{O}(\Xi)_{\mathrm{fin}}\big).$$

THEOREM 5.8. — *Assume that Property* (T) *holds. Fix* (δ_m) *as in Theorem 5.4.*

(i) ρ *is a representation of the Lie algebra* $\mathfrak{g}$ *on* $\mathcal{O}(\Xi)_{\mathrm{fin}}$.

(ii) *The representation* ρ *is irreducible.*

Proof. (i) Since π is a representation of K, for $X, X' \in \mathfrak{k}$,

$$[\rho(X), \rho(X')] = \rho([X, X']).$$

It follows from Proposition 5.7 that, for $X \in \mathfrak{k}, p \in \mathfrak{p}$,

$$[\rho(X), \rho(p)] = \rho([X, p]).$$

It remains to show that, for $p, p' \in \mathfrak{p}$,

$$[\rho(p), \rho(p']] = \rho([p, p'].$$

By Theorem 5.4, $[\rho(E), \rho(F)] = \rho(H)$. Then this follows from Lemma 3.6 in [Brylinski-Kostant,1995]: consider the map

$$\tau : \bigwedge^2 \mathfrak{p} \to \mathrm{End}\big(\mathcal{O}(\Xi)_{\mathrm{fin}},$$

defined by

$$\tau(p \wedge p') = [\rho(p), \rho(p')] - \rho([p, p']).$$

We know that $\tau(E \wedge F) = 0$. It follows that, for $g \in K$,

$$\tau\big(\kappa(g)E \wedge \kappa(g)F\big) = 0.$$

Since the representation κ is irreducible, and E and F are highest and lowest vectors with respect to P, the vector $E \wedge F$ is cyclic in $\bigwedge^2 \mathfrak{p}$ for the action of K. Therefore $\tau \equiv 0$.

(ii) Let $\mathcal{V} \neq \{0\}$ be a $\rho(\mathfrak{g})$-invariant subspace of $\mathcal{O}(\Xi)_{\text{fin}}$. Then $\mathcal{V}$ is $\rho(\mathfrak{k})$-invariant. As $\mathcal{O}(\Xi)_{\text{fin}} = \sum_{m=0}^{\infty} \mathcal{O}_m(\Xi)$ and as the subspaces $\mathcal{O}_m(\Xi)$ are $\rho(\mathfrak{k})$-irreducible, then there exists $\mathcal{I} \subset \mathbb{N}$ ($\mathcal{I} \neq \emptyset$) such that $\mathcal{V} = \sum_{m\in\mathcal{I}} \mathcal{O}_m(\Xi)$. Observe that if $\mathcal{V}$ contains $\mathcal{O}_m(\Xi)$, then it contains $\mathcal{O}_{m+1}(\Xi)$ too. In fact denote by ϕ_m the function in $\mathcal{O}_m(\Xi)$ defined by $\phi_m(w, z) = w^m$. As $\mathcal{D}\phi_m = 0$, it follows that

$$\rho(F)\phi_m = \mathcal{M}\phi_m = \phi_{m+1},$$

and $\rho(F)\phi_m$ belongs to $\mathcal{O}_{m+1}(\Xi)$, therefore $\mathcal{O}_{m+1}(\Xi) \subset \mathcal{V}$. Denote by m_0 the minimum of the m such that $\mathcal{O}_m(\Xi) \subset \mathcal{V}$, then

$$\mathcal{V} = \bigoplus_{m=m_0}^{\infty} \mathcal{O}_m(\Xi).$$

The function $\phi(w, z) = Q(z)^m w^m$ belongs to $\mathcal{O}_m(\Xi)$, and

$$\rho(F)\phi(w, z) = Q(z)^m w^{m+1} - \delta_{m-1} Q\Big(\frac{\partial}{\partial z}\Big) Q(z)^m w^{m-1}.$$

By the Bernstein identity (Proposition 3.1)

$$Q\Big(\frac{\partial}{\partial z}\Big) Q(z)^m = B(m) Q(z)^{m-1},$$

and since $B(m) > 0$ for $m > 0$, it follows that, if $\mathcal{O}_m(\Xi) \subset \mathcal{V}$ with $m > 0$, then $\mathcal{O}_{m-1}(\Xi) \subset \mathcal{V}$. Therefore $m_0 = 0$ and $\mathcal{V} = \mathcal{O}(\Xi)_{\text{fin}}$. $\square$

6. The unitary representation of the Kantor-Koecher-Tits group. — We consider, for a sequence (c_m) of positive numbers, an inner product on $\mathcal{O}(\Xi)_{\text{fin}}$ such that

$$\|\phi\|^2 = \sum_{m=0}^{\infty} \frac{1}{c_m} \|\psi_m\|_m^2,$$

for

$$\phi(w, z) = \sum_{m=0}^{\infty} \psi_m(z) w^m.$$

This inner product is invariant under $K_{\mathbb{R}}$. We assume that Property (T) holds, and we will determine the sequence (c_m) such that this inner product is invariant under the representation ρ restricted to $\mathfrak{g}_{\mathbb{R}}$. We denote by $\mathcal{H}$ the Hilbert space completion of $\mathcal{O}(\Xi)_{\text{fin}}$ with respect to this inner product. We will assume $c_0 = 1$.

The Bernstein polynomial B is of degree 4, and vanishes at 0 and $\alpha_1 = 1-\eta$. Let α_2 and α_3 be the two remaining roots:

$$B(\alpha) = A\alpha(\alpha-\alpha_1)(\alpha-\alpha_2)(\alpha-\alpha_3).$$

(1) $V=\mathbb{C}^n$, $Q(z) = (z_1^2+\cdots+z_n^2)^2$. Then

$$B(\alpha) = A\alpha\Big(\alpha-\frac{1}{2}\Big)\Big(\alpha+\frac{n-4}{4}\Big)\Big(\alpha+\frac{n-2}{4}\Big).$$

$A = 2^4$ if $n\geq 2$, $A = 4^4$ if $n=1$.

(2) $V = (z_1^2+\cdots+z_p^2)(z_{p+1}^2+\cdots+z_{2p}^2)$. Then

$$B(\alpha) = \alpha^2\Big(\alpha+\frac{p-2}{2}\Big)^2.$$

(3) V is simple of rank 4, complexification of $V_{\mathbb{R}} = Herm(4,\mathbb{F})$, $Q(z)=\Delta(z)$, the determinant polynomial. Then

$$B(\alpha) = \alpha\Big(\alpha+\frac{d}{2}\Big)\Big(\alpha+2\frac{d}{2}\Big)\Big(\alpha+3\frac{d}{2}\Big),$$

where $d = \dim_{\mathbb{R}}\mathbb{F}$.

Here are the non zero roots of the Bernstein polynomial:

	η	α_1	α_2	α_3
(1)	$\frac{n}{4}$	$-\frac{n-4}{4}$	$\frac{1}{2}$	$-\frac{n-2}{4}$
(2)	$\frac{p}{2}$	$-\frac{p-2}{2}$	0	$-\frac{p-2}{2}$
(3)	$1+3\frac{d}{2}$	$-3\frac{d}{2}$	$-\frac{d}{2}$	$-2\frac{d}{2}$

THEOREM 6.1. — (i) *The inner product of $\mathcal{H}$ is $\mathfrak{g}_{\mathbb{R}}$-invariant if*

$$c_m = \frac{(\eta+1)_m}{(\eta+\alpha_2)_m(\eta+\alpha_3)_m}\frac{1}{m!}.$$

(ii) *The reproducing kernel of $\mathcal{H}$ is given by*

$$\mathcal{K}(\xi,\xi') = {}_1F_2(\eta+1;\eta+\alpha_2,\eta+\alpha_3;H(z,z')w\overline{w'}),$$

for $\xi=(w,z)$, $\xi'=(w',z')$.

Proof. (i) Recall that

$$\mathfrak{p}_{\mathbb{R}} = \{p \in \mathfrak{p} \mid \beta(p) = p\},$$

where β is the conjugation of $\mathfrak{p}$, we introduced at the end of Section 4. Recall also that

$$\beta(\kappa(g)p) = \kappa\big(\alpha(g)\big)\beta(p).$$

The inner product of $\mathcal{H}$ is $\mathfrak{g}_{\mathbb{R}}$-invariant if and only if, for every $p \in \mathfrak{p}$,

$$\rho(p)^* = -\rho\big(\beta(p)\big).$$

But this is equivalent to the single condition

$$\rho(E)^* = -\rho(F).$$

In fact, assume that this condition is satisfied. Then, for $p = \kappa(g)E$, $(g \in K)$,

$$\rho(p) = \pi(g)\rho(E)\pi(g^{-1}), \quad \rho(p)^* = -\pi(g^{-1})^*\rho(F)\pi(g)^*.$$

Since $\pi(g)^* = \pi\big(\alpha(g)\big)^{-1}$, we get

$$\begin{aligned}\rho(p)^* &= -\pi\big(\alpha(g)\big)\rho(F)\pi\big(\alpha(g^{-1})\big) = -\rho\big(\kappa(\alpha(g))F\big) \\ &= -\rho\big(\kappa(\alpha(g))\beta(E)\big) = -\rho\big(\beta(\kappa(g)E)\big) = -\rho\big(\beta(p)\big).\end{aligned}$$

Finally observe that the vector E is cyclic in $\mathfrak{p}$ for the K-action.

The condition $\rho(E)^* = -\rho(F)$ is equivalent to: for $m \geq 0$, $\phi \in \mathcal{O}_{m+1}(\Xi), \phi' \in \mathcal{O}_m(\Xi)$,

$$\frac{1}{c_{m+1}}(\phi \mid \mathcal{M}^\sigma \phi')_{m+1} = \frac{1}{c_m}\delta_m(\mathcal{D}\phi \mid \phi')_m.$$

Recall that $m_0(dz) = H_0(z)m(dz)$ with

$$H_0(z) = H(z)^{-2\eta},$$

and the norm of $\tilde{\mathcal{O}}_m(V)$ can be written

$$\|\psi\|_m^2 = \frac{1}{a_m}\int_V |\psi(z)|^2 H(z)^{-m-2\eta} m(dz).$$

Then, the required condition of invariance becomes

$$\begin{aligned}&\frac{1}{c_{m+1}a_{m+1}}\int_V \psi(z)\overline{Q(z)\psi'(z)}H(z)^{-(m+1)-2\eta}m(dz) \\ &= \frac{\delta_m}{c_m a_m}\int_V \Big(Q\Big(\frac{\partial}{\partial z}\Big)\psi\Big)(z)\overline{\psi'(z)}H(z)^{-m-2\eta}m(dz).\end{aligned}$$

By integrating by parts:

$$\int_V (Q\Big(\frac{\partial}{\partial z}\Big)\psi)(z)\overline{\psi'(z)}H(z)^{-m-2\eta}m(dz)$$
$$= \int_V \psi(z)\overline{\psi'(z)}\Big(Q\Big(\frac{\partial}{\partial z}\Big)H(z)^{-m-2\eta}\Big)m(dz),$$

and, by the relation

$$Q\Big(\frac{\partial}{\partial z}\Big)H(z)^{-m-2\eta} = B(-m-2\eta)\overline{Q(z)}H(z)^{-(m+1)-2\eta},$$

the condition can be written

$$\frac{1}{c_{m+1}} = \frac{a_{m+1}}{a_m}\delta_m B(-m-2\eta)\frac{1}{c_m}.$$

From Proposition 3.2 it follows that

$$\frac{a_{m+1}}{a_m} = \frac{B(-m-\eta)}{B(-m-2\eta)}.$$

We obtain finally

$$\frac{c_{m+1}}{c_m} = \frac{m+\eta+1}{(m+\eta+\alpha_2)(m+\eta+\alpha_3)(m+1)},$$

and, since $c_0 = 1$,

$$c_m = \frac{(\eta+1)_m}{(\eta+\alpha_2)_m(\eta+\alpha_3)_m}\frac{1}{m!}.$$

(ii) By Theorem 2.5 the reproducing kernel of $\mathcal{H}$ is given by

$$\mathcal{K}(\xi,\xi') = \sum_{m=0}^{\infty} c_m H(z,z')^m w^m \overline{w'}^m$$
$$= {}_1F_2\big(\eta+\alpha_2, \eta+\alpha_3; \eta+1; H(z,z')w\overline{w'}\big),$$

with $\xi = (w,z)$, $\xi' = (w',z')$. $\square$

We will see that the Hilbert space $\mathcal{H}$ is a pseudo-weighted Bergman space. By this we mean that the norm is given by an integral of $|\phi|^2$ with

respect to a weight taking both positive and negative values. The weight involves a Meijer G-function:

$$G(u) = \frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\beta_1+s)\Gamma(\beta_2+s)\Gamma(\beta_3+s)}{\Gamma(\alpha+s)}u^{-s}ds,$$

where $\alpha, \beta_1, \beta_2, \beta_3$ are real numbers, and $c > \sigma = -\inf\{\beta_1, \beta_2, \beta_3\}$. This function is denoted by

$$G(u) = G^{3,0}_{1,3}\Big(x\Big|\begin{matrix}\alpha & & \\ \beta_1 & \beta_2 & \beta_3\end{matrix}\Big)$$

(see for instance [Mathai,1993]). By the inversion formula for the Mellin transform

$$\int_0^\infty G(u)u^{s-1}du = \frac{\Gamma(\beta_1+s)\Gamma(\beta_2+s)\Gamma(\beta_3+s)}{\Gamma(\alpha+s)},$$

for $\operatorname{Re} s > \sigma$, and the integral is absolutely convergent. If the numbers $\beta_1, \beta_2, \beta_3$ are distinct, then

$$G(u) = \varphi_1(u)u^{\beta_1} + \varphi_2(u)u^{\beta_2} + \varphi_3(u)u^{\beta_3},$$

where $\varphi_1, \varphi_2, \varphi_3$ are holomorphic near 0. ($\varphi_1, \varphi_2, \varphi_3$ are ${}_1F_2$ hypergeometric functions.)

The function G may be not positive on $]0,\infty[$, but is positive for u large enough. In fact

$$G(u) \sim \sqrt{\pi}u^\theta e^{-2\sqrt{u}} \quad (u\to\infty),$$

where

$$\theta = \beta_1+\beta_2+\beta_3-\alpha-\tfrac{1}{2}\,.$$

([Paris-Wood,1986], Theorem 3, p.32.)

Now take

$$\alpha = \eta-1,\ \beta_1 = 2\eta-1,\ \beta_2 = 2\eta+a-1,\ \beta_3 = 2\eta+b-1.$$

	α	β_1	β_2	β_3
(1)	$\frac{n}{4}-1$	$\frac{n-2}{2}$	$\frac{n-1}{2}$	$\frac{n-2}{4}$
(2)	$\frac{p}{2}-1$	$p-1$	$p-1$	$\frac{p}{2}$
(3)	$3\frac{d}{2}$	$3d+1$	$5\frac{d}{2}+1$	$2d+1$

The Mellin transform of G vanishes at $-\alpha$, with changing sign. One can check that $-\alpha > \sigma$ in all cases. Therefore there are real values $s > \sigma$ for which the integral

$$\int_0^\infty G(u)u^{s-1}du < 0.$$

This implies that the function G takes negative values on $]0, \infty[$.

THEOREM 6.2. — *For $\phi \in \mathcal{H}$,*

$$\|\phi\|^2 = \int_{\mathbb{C}\times V} |\phi(w,z)|^2 p(z,w)m(dw)m_0(dz),$$

with

$$p(w,z) = CG\big(|w|^2 H(z)\big)H(z).$$

The integral is absolutely convergent.

Proof. We will follow the proof of Theorem 5.7 in [Brylinski,1997].

a) From the proof of Theorem 6.1 it follows that

$$\begin{aligned}\frac{1}{a_m c_m} &= \frac{(2\eta)_m(2\eta+\alpha_2)_m(2\eta+\alpha_3)_m}{(\eta)_m} \\ &= C\frac{\Gamma(2\eta+m)\Gamma(2\eta+\alpha_2+m)\Gamma(2\eta+\alpha_3+m)}{\Gamma(\eta+m)} \\ &= C\int_0^\infty G(u)u^m du.\end{aligned}$$

(One checks that $\sigma < 1$, *i.e.* G is integrable.) By the computation we did for the proof of Theorem 2.6, we obtain, for $\phi(w,z) = w^m\psi(z) \in \mathcal{O}_m$,

$$\int_{\mathbb{C}\times V} |\phi(w,z)|^2 p(z,w)m(dw)m_0(dz) = \|\phi\|^2.$$

Furthermore, if $\phi \in \mathcal{O}_m$, $\phi' \in \mathcal{O}_{m'}$, with $m \neq m'$,

$$\int_{\mathbb{C}\times V} \phi(w,z)\overline{\phi'(w,z)}m(dw)m_0(dz) = 0.$$

It follows that, for $\phi \in \mathcal{O}_{\text{fin}}$,

$$\int_{\mathbb{C}\times V} |\phi(w,z)|^2 p(z,w)m(dw)m_0(dz) = \|\phi\|^2.$$

The computation is justified by the fact that, for $s > \sigma$,

$$\int_0^\infty |G(u)|u^{s-1}du < \infty.$$

b) Let us consider the weighted Bergman space $\mathcal{H}^1$ whose norm is given by

$$\|\phi\|_1^2 = \int_{\mathbb{C}\times V} |\phi(w,z)|^2 |p(w,z)| m(dw) m_0(dz).$$

By Theorem 2.6,

$$\|\phi\|_1^2 = \sum_{m=0}^\infty \frac{1}{c_m^1} \|\psi_m\|_m^2,$$

with

$$\frac{1}{a_m c_m^1} = C \int_0^\infty |G(u)|u^m du.$$

Obviously $c_m^1 \leq c_m$, therefore $\mathcal{H}^1 \subset \mathcal{H}$. We will show that $\mathcal{H} \subset \mathcal{H}^1$. For that we will prove that there is a constant A such that

$$c_m \leq A c_m^1.$$

As observed above there is $u_0 \geq 0$ such that $G(u) \geq 0$, for $u \geq u_0$, and then

$$\int_0^\infty |G(u)|u^m \leq \int_0^\infty G(u)u^m du + 2\int_0^{u_0} |G(u)|u^m du.$$

Hence

$$\frac{1}{c_m^1} \leq \frac{1}{c_m} + 2a_m u_0^m \int_0^{u_0} |G(u)| du.$$

By the formula we gave at the beginning of a), the sequence $a_m c_m u_0^m$ is bounded. Therefore there is a constant A such that

$$\frac{1}{c_m^1} \leq A \frac{1}{c_m},$$

and this implies that $\mathcal{H} \subset \mathcal{H}_1$. $\square$

Let $\tilde{G_{\mathbb{R}}}$ be the connected and simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and denote by $\tilde{K_{\mathbb{R}}}$ the subgroup of $\tilde{G_{\mathbb{R}}}$ with Lie algebra $\mathfrak{k}_{\mathbb{R}}$. It is a covering of $K_{\mathbb{R}}$. We denote by $s : \tilde{K_{\mathbb{R}}} \to K_{\mathbb{R}}, g \mapsto s(g)$ the canonical surjection.

Theorem 6.3. — (i) *There is a unique unitary irreducible representation $\tilde{\pi}$ of $\tilde{G_{\mathbb{R}}}$ on $\mathcal{H}$ such that $d\tilde{\pi} = \rho$. And, for all $k \in \tilde{K_{\mathbb{R}}}$, $\tilde{\pi}(k) = \pi(s(k))$.*

(ii) *The representation $\tilde{\pi}$ is spherical.*

Proof. (i) Notice that if the operators $\rho(E+F)$ and $\rho(i(E-F))$ are skew-symmetric, then for each $p \in \mathfrak{p}_{\mathbb{R}}$, the operator $\rho(p)$ is skew-symmetric. In fact, since the $\mathfrak{sl}_2$-triple (E, F, H) is strictly normal (see [Sekiguchi,1987]), which means that $H \in i\mathfrak{k}_{\mathbb{R}}, E+F \in \mathfrak{p}_{\mathbb{R}}, i(E-F) \in \mathfrak{p}_{\mathbb{R}}$, and since $\mathfrak{p} = \mathcal{U}(\mathfrak{k})E$, hence $\mathfrak{p}_{\mathbb{R}} = \mathcal{U}(\mathfrak{k}_{\mathbb{R}})(E + F) + \mathcal{U}(\mathfrak{k}_{\mathbb{R}})(i(E - F))$, and the assertion follows.

Now, by Nelson's criterion, it is enough to prove that the operator $\rho(\mathcal{L})$ is essentially self-adjoint where $\mathcal{L}$ is the Laplacian of $\mathfrak{g}_{\mathbb{R}}$. Let's consider a basis $\{X_1, \ldots, X_k\}$ of $\mathfrak{k}_{\mathbb{R}}$ and a basis $\{p_1, \ldots, p_l\}$ of $\mathfrak{p}_{\mathbb{R}}$, orthogonal with respect to the Killing form. As $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} + \mathfrak{p}_{\mathbb{R}}$ is the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$, then the Laplacian and the Casimir operators of $\mathfrak{g}_{\mathbb{R}}$ are given by

$$\mathcal{L} = X_1^2 + \ldots + X_k^2 + p_1^2 + \ldots + p_l^2,$$
$$\mathcal{C} = X_1^2 + \ldots + X_k^2 - p_1^2 - \ldots - p_l^2.$$

It follows that $\mathcal{L} = 2(X_1^2+\ldots+X_k^2)-\mathcal{C}$ and $\rho(\mathcal{L}) = 2\rho(X_1^2+\ldots+X_k^2)-\rho(\mathcal{C})$. Since $\rho(X_1^2 + \ldots + X_k^2) = d\pi(X_1^2 + \ldots + X_k^2)$ and as π is a unitary representation of $K_{\mathbb{R}}$, hence the image $\rho(X_1^2 + \ldots + X_k^2)$ of the Laplacian of $\mathfrak{k}_{\mathbb{R}}$ is essentially self-adjoint. Moreover, since the dimension of $\mathcal{O}(\Xi)_{\text{fin}}$ is countable, then the commutant of ρ, which is a division algebra over $\mathbb{C}$, has a countable dimension too, and is equal to $\mathbb{C}$ (see [Cartier,1979], p.118). It follows that $\rho(\mathcal{C})$ is scalar. We deduce that $\rho(\mathcal{L})$ is essentially self-adjoint and that the irreducible representation ρ of $\mathfrak{g}_{\mathbb{R}}$ integrates to an irreducible unitary representation of $\tilde{G_{\mathbb{R}}}$, on the Hilbert space $\mathcal{H}$.

(ii) The space $\mathcal{O}_0(\Xi)$ reduces to the constant functions which are the K-fixed vectors. □

We don't know whether the representation $\tilde{\pi}$ goes down to a representation of a real Lie group $G_{\mathbb{R}}$ with $K_{\mathbb{R}}$ as a maximal compact subgroup.

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